THE

THEORY OF EQUATIONS

WITH AN

INTRODUCTION TO THE THEORY OF BINARY ALGEBRAIC FORMS

BY THE LATE

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PREFACE

We have endeavoured in the present work to combine some of the modern developments of Higher Algebra with the subjects usually included in works on the Theory of Equations. Of the two volumes into which this work is now divided the first eleven Chapters of Vol. I contain all the propositions ordinarily found in elementary treatises on the subject. In these Chapters we have not hesitated to employ the more modern notation wherever it appeared that greater simplicity or comprehensiveness could be thereby obtained. We have thought it desirable also to add at the close of this volume a short Chapter on *Complex Numbers* and the *Complex Variable*.

Regarding the algebraical and the numerical solution of equations as essentially distinct problems, we have purposely omitted in Chap. VI numerical examples in illustration of the modes of solution there given of the cubic and biquadratic equations. Such examples do not render clearer the conception of an algebraical solution ; and, for practical purposes, the algebraical formula may be regarded as almost useless in the case of equations of a degree higher than the second.

In the treatment of Elimination and Linear Transformation, as well as in the more advanced treatment of Symmetric Functions, a knowledge of Determinants is indispensable. We have found it necessary, therefore, to give, at the opening of the second volume, which contains the subjects included under the title of *Modern Higher Algebra*, a Chapter on Determinants. It has been our aim to make this Chapter as simple and intelligible as possible to the beginner ; and at the same time to omit no proposition which might be found useful in the application of this calculus. For many of the examples in this Chapter, as well as in other parts of the work, we are indebted to kindness of Mr. Cathcart, Fellow of Trinity College.

We have approached the consideration of the Covariants and Invariants through the medium of the functions of the differences of the roots of equations. This appears to be the simplest and most attractive mode of presenting the subject to beginners, and has the advantage, as will be seen, of enabling us to express irrational covariants rationally in terms of the roots. We have attempted at the same time to show how this mode of treatment may be brought into harmony with the more general problem of the linear transformation of algebraic forms.

On the works which have afforded us assistance in the more elementary part of the subject, we wish to mention particularly the *Traité d'Algebre* of M. Bertrand, and the writings of the late Professor Young of Belfast, which have contributed so much to extend and simplify the analysis and solution of numerical equations.

In the more advanced portions of the subject we are indebted mainly, among published works, to the *Lessons Introductory to the Modern Higher Algebra* of Dr. Salmon, and the *Théorie der binären algebraischen Formen* of Clebsch ; and in some degree to the *Théorie des Formes binaires* of the Chev. F. Faa' De Bruno. We must record also our obligation in this department of the subject to Mr. Michael Roberts, from whose papers in the *Quarterly Journal* and other periodicals, and from whose professorial lectures in the University of Dublin, very great assistance has been derived. Many of the examples also are taken from Papers set by him at the University Examinations.

In connection with various parts of the subject several other works have been consulted, among which may be mentioned the treatises on Algebra by Serret, Meyer Hirsch and Rubini, and papers in the mathematical journals by Boole, Cayley, Hermite, and Sylvester.

We have added also in this and preceding edition to what was contained in the earlier editions of this work, a new Chapter on the Theory of Substitutions and Groups. Our aim has been to give here, within as narrow limits as possible, an account of the subject which may be found useful by students as an introduction to those fuller and more systematic works which are specially devoted to this department of Algebra. The works which have afforded us most assistance in the preparation of this Chapter are—Serret's *Cours d' Algebre supérieure* ; *Traité des Substitutions et des Equations algébriques* by M. Camile Jordan (Paris 1870) ; Netto's *Substitutionentheorie und ihre Anwendung auf die Algebra* (Leipzig, 1882), of which there is an English translation by F.N. Cole (Ann. Arbor, Mich., 1892) ; and *Leçons sur la Résolution algébrique des Equations*, by M. H. Vogt (Paris, 1895).

TRINITY COLLEGE DUBLIN, W.S.B.
May, 1904.

PREFACE TO SEVENTH EDITION OF VOLUME I

HAVING been deprived of Dr. Panton's co-operation, I have thought it inadvisable to make any change in this volume by the introduction of new matter, as I wished as far as possible to preserve the original design, and also the limits fixed for this volume as indicated in the Preface to previous editions. In coming to this decision I have been influenced to a great degree by the favourable eception already given to this work.

June 18th, 1912. W.S.B.

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THEORY OF EQUATIONS

INTRODUCTION

1. Definitions. Any mathematical expression involving a quantity is called a *function* of that quantity

We shall be employed mainly with such algebraical functions as are *rational* and *integral*. By a *rational* function of a quantity is meant one which contains that quantity in a rational form only: that is, a form free from fractional indices or radical signs. By an *integral* function of a quantity is meant one in which the quantity enters in an integral form only; that is, never in the denominator of a fraction. The following expression, for example, in which n is a positive integer, is a *rational* and integral *algebraical function* of x :—

$$ax^n + bx^{n-1} + cx^{n-2} + \dots\dots + kx + l$$

It is to be observed that this definition has reference to the quantity x only, of which the expression is regarded as a function. The several co-efficients a, b, c, etc., may be irrational or fractional, and the function still remains rational and integral in x.

A function of x is represented for brevity by $F(x)$, $f(x)$, $\varphi(x)$, or some such symbol.

The name *polynomial* is given to the algebraical function to express the fact that it is constituted of a number of terms containing different powers of x connected by the signs plus or minus. For certain values of x regarded as variable one polynomial may become equal to another differently constituted. The algebraical expression of such a relation is called an *equation*; and any value of x which satisfies this equation is called a *root* of the equation. The determination of all possible roots constitutes the complete *solution of the equation.*

It is obvious that, by bringing all the terms to one side, we may arrange any equation according to descending powers of x in the following manner :—

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_{n-1}x + a_n = 0$$

The highest power of x in this equation being n, it is said to be an equation of the n^{th} *degree* in x. For such an equation we shall, in general, employ the form here written. The suffix attached to the letter a indicates the power of x which each co-efficient accompanies, the sum of the exponent of x and the suffix of a being equal to n for

each term. An equation is not altered if all its terms be divided by any quantity We may thus, if we please, dividing by a_0, make the co efficient of x^n in the above equation equal to unity. It will often be found convenient to make this supposition ; and in such cases the equation will be written in the form

$$x^n + p_1 x^{n-1} + p_2 x^{n-2} + \dots + p_{n-1} x + p_n = 0.$$

An equation is said to be *complete* when it contains terms involving x in all its powers from n to 0, and *incomplete* when some of the terms are absent ; or, in other words, when some of the co-efficients p_1, p_2, etc., are equal to zero. The term p_n, which does not contain x, is called the *absolute term.* An equation is *numerical* or *algebraical* according as its co-efficients are numbers or algebraical symbols.

2. Numerical and Algebraical Equations. In many researches in both mathematical and physical science the final mathematical problem presents itself in the form of an equation on whose solution that of the problem depends. It is natural, therefore, that the attention of mathematicians should have been at an early stage in the history of the science directed towards inquiries of this nature. The science of the Theory of Equations, as it now stands, has grown out of successive attempts of mathematicians to discover general methods for the solution of equations of any degree. When the co-efficients of an equation are given numbers, the problem is to determine a numerical value or perhaps several different numerical values, which will satisfy the equation. In this branch of the science very great progress has been made ; and the best methods hitherto advanced for the discovery, either exactly or approximately, of the numerical values of the roots will be explained in their proper places in this work.

Equal progress has not been made in the general solution of equations whose co-efficients are algebraical symbols. The student is aware that the root of an equation of the second degree, whose co-efficients are such symbols, may be expressed in terms of these co-efficients in a general formula ; and that the numerical roots of any particular numerical equation may be obtained by substituting in this formula the particular numbers for the symbols. It was natural to inquire whether it was possible to discover any such formula for the solution of equations of higher degrees. Such results have been attained in the case of equations of the third and fourth degrees. It will be shown that in certain cases these formulas fail to supply the solution of a numerical equation by substitution of the

numerical co-efficients for the general symbols, and are, therefore, in this respect inferior to the corresponding algebraical solution of the quadratic.

Many attempts have been made to arrive at similar general formulas for equations of the fifth and higher degrees; but it may now be regarded as established by the researches of modern analysis that it is not possible by means of radical signs, and other signs of operation employed in common algebra, to express the root of an equation of the fifth or any higher degree in terms of the co-efficients.

3. Polynomials. From the preceding observations it is plain that one important object of the science of the Theory of Equations is the discovery of those values of the quantity x regarded as variable which give to the polynomial $f(x)$ the particular value zero. In attempting to discover such values of x, we shall be led into many inquiries concerning the values assumed by the polynomial for other values of the variable. We shall, in fact, see in the next chapter that, corresponding to a continuous series of values of x varying from an infinitely great negative quantity $(-\infty)$ to an infinitely great positive quantity $(+\infty)$, $f(x)$ will assume also values continuously varying. The study of such variations is a very important part of the theory of polynomials. The general solution of numerical equations is, in fact, a tentative process; and by examining the values assumed by the polynomial for certain arbitrarily assumed values of the variable, we shall be led, if not to the root itself, at least to an indication of the neighbourhood in which it exists, and within which our further approximation must be carried on.

A polynomial is sometimes called a *quantic.* It is convenient to have distinct names for the quantics of various successive degrees The terms *quadratic* (or *quadric*), *cubic*, *biquadratic* (or *quartic*), *quintic*, *sextic* etc., are used to represent quantics of the 2nd, 3rd, 4th, 5th, 6th, etc., degrees; and the equations obtained by equating these quantics to zero are called *quadratic*, *cubic*, *biquadratic*, etc., *equations*, respectively.

CHAPTER I

GENERAL PROPERTIES OF POLYNOMIALS

4. In tracing the changes of values of a polynomial corresponding to changes in the variable, we shall first inquire what terms in the polynomial are most important when values very great or very small are assigned to x. This inquiry will form the subject to the present and succeeding Articles.

Writing the polynomial in the form

$$a_0x^n\left\{1+\frac{a_1}{a_0}\frac{1}{x}+\frac{a_2}{a_0}\frac{1}{x^2}+\ldots+\frac{a_{n-1}}{a_0}\frac{1}{x^{n-1}}+\frac{a_n}{a_0}\frac{1}{x^n}\right\},$$

it is plain that its value tends to become equal to a_0x^n as x tends towards ∞. The following theorem will determine a quantity such that the substitution of this, or of any greater quantity, for x will have the effect of making the term a_0x^n exceed the sum of all the others. In what follows we suppose a_0 to be positive; and in general in the treatment of polynomials and equations the highest term is supposed to be written with the positive sign.

Theorem. *If in the polynomial*

$$a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\ldots+a_{n-1}x+a_n$$

the value $\frac{a_k}{a_0}+1$, *or any greater value, be substituted for* x, *where* a_k *is that one of the co-efficients* $a_1, a_2, \ldots, a_n$ *whose numerical value is greatest, irrespective of sign, the term containing the highest power of* x *will exceed the sum of all the terms which follow*

The inequality

$$a_0x^n>a_1x^{n-1}+a_2x^{n-2}+\ldots+a_{n-1}x+a_n$$

is satisfied by any value of x which makes

$$a_0x^n>a_k(x^{n-1}+x^{n-2}+\ldots+x+1),$$

where a_k is the greatest among the co-efficients $a_1, a_2, \ldots, a_{n-1}, a_n$ without regard to sign. Summing the geometric series within the brackets, we have

$$a_0x^n>a_k\frac{x^n-1}{x-1}, \text{ or } x^n>\frac{a_k}{a_0(x-1)}(x^n-1),$$

which is satisfied if $a_0(x-1)$ be $>$ or $=a_k$,

that is $\qquad x>$ or $=\frac{a_k}{-}+1.$

The theorem here proved is useful in supplying, when the co-efficients of the polynomial are given numbers, a number such that when x receives values nearer to $+\infty$, the polynomial will preserve constantly a positive sign. If we change the sign of x, the first term will retain its sign if n be even, and will become negative if n be odd; so that the theorem also supplies a negative value of x, such that for any value nearer to $-\infty$ the polynomial will retain constantly a positive sign if n be even, and a negative sign if n be odd. The constitution of the polynomial is, in general, such that limits much nearer to zero than those here arrived at can be found beyond which the function preserves the same sign; for in the above proof we have taken the most unfavourable case, *viz.*, that in which all the co-efficients except the first are negative, and each equal to a_k; whereas in general the co-efficients may be positive, negative, or zero. Several theorems, having for their object the discovery of such closer limits, will be given in a subsequent chapter.

5. We now proceed to inquire what is the most important term in a polynomial when the value of x is indefinitely diminished; and to determine a quantity such that the substitution of this, or of any smaller quantity, for x will have the effect of giving such term the preponderance.

Theorem. *If in the polynomial*

$$a_0x^n+a_1x^{n-1}+\ldots+a_{n-1}x+a_n$$

the value $\dfrac{a_n}{a_n+a_k}$, *or any smaller value, be substituted for x, where a_k is the greatest co-efficient exclusive of a_n, the term a_n will be numerically greater than the sum of all the others.*

To prove this, let $x=\dfrac{1}{y}$; then by the theorem of Art. 4, a_k being now the greatest among the co-efficients $a_0, a_1, \ldots, a_{n-1}$, without regard to sign, the value $\dfrac{a_k}{a_n}+1$, or any greater value of y, will make

$$a_ny^n>a_{n-1}y^{n-1}+a_{n-2}y^{n-2}+\ldots+a_1y+a_0.$$

that is, $$a_n>a_{n-1}\frac{1}{y}+a_{n-2}\frac{1}{y^2}+\ldots\ldots+a_0\frac{1}{y^n}$$

hence the value $\dfrac{a_n}{a_n+a_k}$, or any less value of x, will make

$$a_n>a_{n-1}x+a_{n-2}x^2+\ldots+a_0x^n.$$

This proposition is often stated in a different manner, as follows :—*Values so small may be assigned to x as to make the polynomial*

$$a_{n-1}x+a_{n-2}x^2+\dots+a_0x^n$$

less than any assigned quantity.

This statement of the theorem follows at once from the above proof, since a_n may be taken to be the assigned quantity.

There is also another useful statement of the theorem, as follows :—*When the variable x receives a very small value, the sign of the polynomial*

$$a_{n-1}x+a_{n-2}x^2+\dots+a_0x^n$$

is the same as the sign of its first term $a_{n-1}x$.

This appears by writing the expression in the form

$$x\{a_{n-1}+a_{n-2}x+\dots+a_0x^{n-1}\} ;$$

for when a value sufficiently small is given to x, the numerical value of the term a_{n-1} exceeds the sum of the other terms of the expression within the brackets. and the sign of that expression will consequently depend on the sign of a_{r-1}

6. Change of Form of a Polynomial corresponding to an increase or diminution of the Variable. Derived Functions. We shall now examine the form assumed by the polynomial when $x+h$ is substituted for x. If, in what follows, h be supposed essentially positive, the resulting form will correspond to an increase of the variable ; and the form corresponding to a diminution of x will be obtained from this by changing the sign of h in the result.

When x is changed to $x+h$, $f(x)$ becomes $f(x+h)$, or

$$a_0(x+h)^n+a_1(x+h)^{n-1}+a_2(x+h)^{n-2}+\dots+a_{n-1}(x+h)+a_n.$$

Let each term of this expression be expanded by the binomial theorem and the result arranged according to ascending powers of h. We then have

$$\begin{aligned}&a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\dots+a_{n-2}x^2+a_{n-1}x+a_n\\ &\quad+h\{na_0x^{n-1}+(n-1)\,a_1x^{n-2}+(n-2)\,a_2x^{n-3}+\dots+2a_{n-2}x+a_{n-1}\}\\ &\quad+\frac{h^2}{1.2}\{n(n-1)\,a_0x^{n-2}+(n-1)(n-2)\,a_1x^{n-3}+\dots+2a_{n-2}\}\\ &\quad+\dots\dots\dots\dots\dots\\ &\quad+\frac{h^n}{1.2.3\dots n}\{n.n-1\dots2.1\}a_0.\end{aligned}$$

It will be observed that the part of this expression independent of h is $f(x)$ (a result obvious *a priori*), and that the successive co-efficients of the different powers of h are functions of x of degrees diminishing by unity. It will be further observed that the co-efficient of h may be derived from $f(x)$ in the following manner :—Let each term in $f(x)$ be multiplied by the exponent of x in that term, and let the exponent of x in the term be diminished by unity, the sign being

retained; the sum of all the terms of $f(x)$ treated in this way will constitute a polynomial of dimensions one degree lower than those of $f(x)$. This polynomial is called the *first derived function of* $f(x)$. It is usual to represent this function by the notation $f'(x)$. The co-efficient of $\frac{h^2}{1.2}$ may be derived from $f'(x)$, by a process the same as that employed in deriving $f'(x)$ from $f(x)$, or by the operation twice performed on $f(x)$. This co-efficient is represented by $f''(x)$, and is called the *second derived function* of $f(x)$. In like manner the succeeding co-efficients may all be derived by successive operations of this character; so that, employing the notation here indicated, we may write the result as follows :—

$$f(x+h)=f(x)+f'(x)h+\frac{f''(x)}{1\,.\,2}h^2+\frac{f'''(x)}{1.2.3}h^3+\ldots+a_0h^n.$$

It may be observed that, since the interchange of x and h does not alter $f(x+h)$, the expansion may also be written in the form

$$f(x+h)=f(h)+f'(h)x+\frac{f''(h)}{1\,.\,2}x^2+\frac{f'''(h)}{1\,.\,2\,.\,3}x^3+\ldots+a_0x^n$$

We shall in general employ the notation here explained; but on certain occasions when it is necessary to deal with derived functions beyond the first two or three, it will be found more convenient to use suffixes instead of the accents here employed. The expansion will then be written as follows :—

$$f(x+h)=f(x)+f_1(x)h+f_2(x)\frac{h^2}{1\,.\,2}+\ldots+f_r(x)\frac{h^r}{1\,.\,2\,.\,3\ldots r}+\ldots$$

Example

Find the result of substituting $x+h$ for x in the polynomial

$$4x^3+6x^2-7x+4.$$

Here

$$f(x)=4x^3+6x^2-7x+4,$$
$$f'(x)=12x^2+12x-7,$$
$$f''(x)=24x+12,$$
$$f'''(x)=24;$$

and the result is

$$4x^3+6x^2-7x+4+(12x^2+12x-7)h+(24x+12)\frac{h^2}{1\,.\,2}+24\frac{h^3}{1\,.\,2\,.\,3}.$$

The student may verify this result by direct substitution.

7. Continuity of a Rational Integral Function of x. If in a rational and integral function $f(x)$ the value of x be made to vary, by indefinitely small increments, from one quantity a to a greater quantity b, we proceed to prove that $f(x)$ at the same time varies also by indefinitely small increments; in other words, that $f(x)$ *varies continuously with* x

Let x be increased from a to $a+h$. The corresponding increment of $f(x)$ is

$$f(a+h)-f(a);$$

and this is equal, by Art. 6, to

$$f'(a)h+f''(a)\frac{h^2}{1\,.\,2}+\ldots+a_0h^n,$$

in which expression all the co-efficients $f'(a)$, $f''(a)$, etc., are finite quantities. Now, by the theorem of Art. 5, this latter expression may, by taking h small enough, be made to assume a value less than any assigned quantity, so that the difference between $f(a+h)$ and $f(a)$ may be made as small as we please, and will ultimately vanish with h. The same is true during all stages of the variation of x from a to b; thus the continuity of the function $f(x)$ is established.

It is to be observed that it is not here proved that $f(x)$ *increases* continuously from $f(a)$ to $f(b)$. It may either increase or diminish, or at one time increase, and at another diminish; but the above proof shows that it cannot pass *per saltum* from one value to another; and that, consequently, amongst the values assumed by $f(x)$ while x increases continuously from a to b must be included all values between $f(a)$ and $f(b)$. The sign of $f'(a)$ will determine whether $f(x)$ is increasing or diminishing; for it appears by Art. 5 that when h is small enough the sign of the total increment will depend on that of $f'(a)h$. We thus observe that *when $f'(a)$ is positive $f(x)$ is increasing with x; and when $f'(a)$ is negative $f(x)$ is diminishing as x increases.*

8. Form of the Quotient and Remainder when a Polynomial is divided by a Binomial. Let the quotient, when

$$a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\ldots+a_{n-1}x+a_n$$

is divided by $x-h$, be

$$b_0x^{n-1}+b_1x^{n-2}+\ldots+b_{n-2}x+b_{n-1}.$$

This we shall represent by Q, and the remainder by R. We have then the following equation:—

$$f(x)\equiv(x-h)Q+R.$$

The meaning of this equation is, that when Q is multiplied by $x-h$, and R added, the result must be *identical*, term for term, with $f(x)$. In order to distinguish equations of the kind here explained from equations which are not identities, it will often be found convenient to use the symbol here employed in place of the usual symbol of equality. The right-hand side of the identity is

$$b_0x^n+\left.\begin{matrix}b_1\\-hb_0\end{matrix}\right\}x^{n-1}+\left.\begin{matrix}b_2\\-hb_1\end{matrix}\right\}x^{n-2}+\ldots+\left.\begin{matrix}b_{n-1}\\-hb_{n-2}\end{matrix}\right\}x+\begin{matrix}R\\-hb_{n-1}.\end{matrix}$$

Equating the co-efficients of x on both sides, we get the following series of equations to determine $b_0, b_1, b_2, \ldots\ldots, b_{n-1}, R$:—

$$\begin{aligned}
b_0 &= a_0, \\
b_1 &= b_0h + a_1, \\
b_2 &= b_1h + a_2, \\
b_3 &= b_2h + a_3, \\
&\cdots\cdots \\
b_{n-1} &= b_{n-2}h + a_{n-1}, \\
R &= b_{n-1}h + a_n.
\end{aligned}$$

These equations supply a ready method of calculating in succession the co-efficients b_0, b_1, etc. of the quotient, and the remainder R. For this purpose, we write the series of operations in the following manner :—

$$\begin{array}{llllll}
a_1, & a_2, & a_3 & \ldots\ldots & a_{n-1}, & a_n, \\
b_0h, & b_1h, & b_2h, & \ldots\ldots & b_{n-2}h, & b_{n-1}h, \\ \hline
b_1, & b_2, & b_3, & \ldots\ldots & b_{n-1}, & R.
\end{array}$$

In the first line are written down the successive co-efficients of $f(x)$. The first term in the second line is obtained by multiplying a_0 (or b_0, which is equal to it) by h. The product b_0h is placed under a_1, and then added to it in order to obtain the term b_1 in the third line. This term, when obtained, is multiplied in its turn by h, and placed under a_2. The product is added to a_2 to obtain the second figure b_2 in the third line. The repetition of this process furnishes in succession all the co-efficients of the quotient, the last figure thus obtained being the remainder. A few examples will make this plain.

Examples

1. Find the quotient and remainder when $3x^4-5x^3+10x^2+11x-61$ is divided by $x-3$.

The calculation is arranged as follows :—

$$\begin{array}{rrrrr}
3 & -5 & 10 & 11 & -61. \\
 & 9 & 12 & 66 & 231. \\ \hline
 & 4 & 22 & 77 & 170.
\end{array}$$

Thus the quotient is $3x^3+4x^2+22x+77$, and the remainder 170.

2. Find the quotient and remainder when x^3+5x^2+3x+2 is divided by $x-1$. [*Ans.* $Q=x^2+6x+9$, $R=11$.

3. Find Q and R when $x^5-4x^4+7x^3-11x-13$ is divided by $x-5$.

N.B.—When any term in a polynomial is absent, care must be taken to supply the place of its co-efficient by zero in writing down the co-efficients of $f(x)$. In this example, therefore, the series in the first line will be

$$1 \quad -4 \quad 7 \quad 0 \quad -11 \quad -13.$$

[*Ans.* $Q=x^4+x^3+12x^2+60x+289$; $R=1432$.

4. Find Q and R when $x^9+3x^7-15x^2+2$ is divided by $x-2$.

[*Ans.* $Q=x^8+2x^7+7x^6+14x^5+28x^4+56x^3+112x^2+209x+418$; $R=838$.

5. Find Q and R when $x^5+x^2-10x+113$ is divided by $x+4$.

[*Ans.* $Q=x^4-4x^3+16x^2-63x+242$; $R=-855$.

9. Tabulation of Functions. The operation explained in the preceding Article affords a convenient practical method of calculating the numerical value of a polynomial whose co-efficients are given numbers when any number is substituted for x. For the equation

$$f(x)\equiv(x-h)\ Q+R,$$

since its two members are identically equal, must be satisfied when any quantity whatever is substituted for x. Let $x=h$, then $f(h)=R$, $x-h$ being $=0$, and Q remaining finite. Hence the result of substituting h for x in $f(x)$ is the remainder when $f(x)$ is divided by $x-h$, and can be calculated rapidly by the process of the last Article.

For example, the result of substituting 3 for x in the polynomial of Ex. 1, Art. 8, *viz.*,

$$3x^4-5x^3+10x^2+11x-61,$$

is 170, this being the remainder after division by $x-3$. The student can verify this by actual substitution.

Again, the result of substituting -4 for x in

$$x^5+x^2-10x+113$$

is -855, as appears from Ex. 5, Art. 8. We saw in Art. 7 that as x receives a continuous series of values increasing from $-\infty$ to $+\infty$, $f(x)$ will pass through a corresponding continuous series. If we substitute in succession for x, in a polynomial whose co-efficients are given numbers, a series of numbers such as

$$\ldots -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, \ldots,$$

and calculate the corresponding values of $f(x)$, the process may be called the *tubulation of the function*.

Examples

1. Tabulate the trinomial $2x^2+x-6$, for the following values of x :—

$$-4,\ -3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3,\ 4.$$

Values of x,	-4	-3		-1	0	1		3	4
,, ,, $f(x)$,	22	9	0			-3		15	30

2. Tabulate the polynomial $10x^3-17x^2+x+6$ for the same values of x.

Values of x,	-4	-3	-2	-1	0	1	2	3	4
,, ,, $f(x)$.	-910	-420	-144	-22	6	0	20	126	378

10. Graphic Representation of a Polynomial. In investigating the changes of a function $f(x)$ consequent on any series of changes in the variable which it contains, it is plain that great advantage will be derived from any mode of representation which renders

possible a rapid comparison with one another of the different values which the function may assume. In the case where the function in question is a polynomial with numerical co-efficients, to any assumed value of x will correspond one definite value of $f(x)$. We proceed to explain a mode of graphic representation by which it is possible to exhibit to the eye the several values of $f(x)$ corresponding to the different values of x.

Let two right lines OX, OY (Fig. 1) cut one another at right angles, and be produced indefinitely in both directions. These lines are called the *axis of x* and *axis of y*, respectively. Lines, such as OA, measured on the axis of x at the right-hand side of O, are regarded as positive; and those, such as OA', measured at the left-hand side, as negative. Lines parallel to OY which are above XX' such as AP or $B'Q'$, are positive; and those below it, such as AT or $A'P'$ are negative. These conventions are already familiar to the student acquainted with Trigonometry.

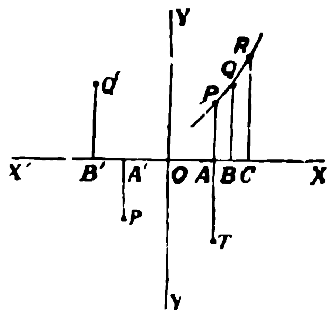

Fig. 1.

Any arbitrary length may now be taken on OX as unity, and any number positive or negative will be represented by a line measured on XX'; the series of numbers increasing from 0 to $+\infty$ in the direction OX, and diminishing from 0 to $-\infty$ in the direction OX'. Let any number m be represented by OA; calculate $f(m)$; from A draw AP parallel to OY to represent $f(m)$ in magnitude on the same scale as that on which OA represents m, and to represent by its position above or below the line OX the sign of $f(m)$. Corresponding to the different values of m represented by OA, OB, OC, etc., we shall have a series of points P, Q, R, etc., which when we suppose the series of values of m indefinitely increased so as to include all numbers between $-\infty$ and $+\infty$, will trace out a continuous curved line. This curve will, by the distance of its several points from the line OX, exhibit to the eye the several values of the function $f(x)$.

The process here explained is also called *tracing the function* $f(x)$. The student acquainted with analytic geometry will observe that it is equivalent to tracing the plane curve whose equation is $y=f(x)$.

In the practical application of this method it is well to begin by laying down the points on the curve corresponding to certain small integral values of x, positive and negative. It will then in general be

possible to draw through these points a curve which will exhibit the progress of the function, and give a general idea of its character. The accuracy of the representation will of course increase with the number of points determined between any two given values of the variable. When any portion of the curve between two proposed limits has to be examined with care, it will often be necessary to substitute values of the variable separated by smaller intervals than unity. The following examples will illustrate these principles.

Examples

1. Trace the trinomial $2x^2+x-6$.

The unit of length taken is one-sixth of the line OD in Fig. 2.

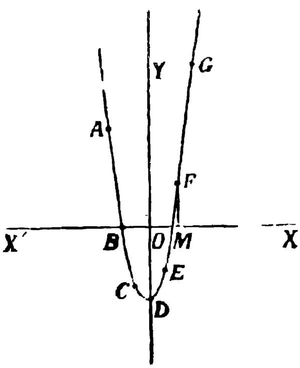

Fig. 2.

In Ex. 1, Art. 9, the values of $f(x)$ are given corresponding to the integral values of x from -4 to $+4$, inclusive.

By means of these values we obtain the positions of nine points on the curve; seven of which, A, B, C, D, E, F, G, are here represented; the other two corresponding to values of $f(x)$ which lie out of the limits of the figure.

The student will find it a useful exercise to trace the curve more minutely between the points C and E in the figure, viz., by calculating the values of $f(x)$ corresponding to the values of x between -1 and 1 separated by small intervals, say of one-tenth, as is done in the following example.

2. Trace the polynomial

$$10x^3-17x^2+x+6.$$

This is already tabulated in Art. 9 for values of x between -4 and 4.

It may be observed, as an exercise on Art. 4, that this function retains positive values for all positive values of x greater than 2·7, and negative values for all values of x nearer to $-\infty$ than $-2·7$. The curve will, then if it cuts the axis of x at all, cut it at a point (or points corresponding to some value (or values) of x between $-2·7$ and $+2·7$; so that if our object is to determine, or appoximate to, the positions of the roots of the equation $f(x)=0$, the tabulation may be confined to the interval between $-2·7$ and $2·7$.

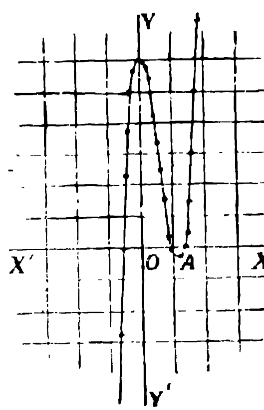

Fig. 3.

This is a case in which the substitution of integral values only of x gives very little help towards the tracing of the curve, and where, consequently, smaller intervals have to be examined. We give the tabulation of the function for intervals of one-tenth between the integers -1, 0; 0, 1; 1, 2. From these values the positions of the corresponding points on the curve may be approximately ascertained, and the curve traced as in Fig

Values of x	-1	$-·9$	$-·8$	$-·7$	$-·6$	$-·5$	$-·4$	$-·3$	$-·2$	$-·1$	
„ „ $f(x)$	-22	-15 96	$-10·8$	-6 46	-2 88	0	2 24	3·9	5 04	5 72	
Values of x	0	·1	·2	·3	·4	·5	6	·7	·8	·9	
„ „ $f(x)$	6	5·94	5·6	5·04	4·32	3·5	2·64	1 8	1·04	·42	
Values of x	1	1·1	1·2	1·3	1·4	1·5	1·6	1·7	1·8	1·9	2
„ „ $f(x)$	0	$-·16$	0	·54	1·52	3	5·04	7·7	11 04	15·12	20

The curve traced in Ex. 1 cuts the axis of x in two points (a number equal to the degree of the polynomial) : in other words, there are two values of x for which the value of the given polynomial is zero ; these are the roots of the equation $2x^2+x-6=0$, *viz.*, -2, and 1·5. Similarly, the curve traced in Ex. 2 cuts the axis in three points, viz. the points corresponding to the roots of the cubic equation $10x^3-17x^2-x+6=0$. The curve representing a given polynomial may not cut the axis of x at all, or may cut it in a number of points less than the degree of the polynomial. Such cases correspond to the imaginary roots of equations, as will appear more fully in the next chapter. For example, the curve which represents the polynomial $2x^2+x+2$ will, when traced, lie entirely above the axis of x ; in fact, since this function differs from the function of Ex. 1, only by the addition of the constant quantity 8, each value of $f(x)$ is obtained by adding 8 to the previously calculated value, and the entire curve can be obtained by simply supposing the previously traced curve to be moved up parallel to the axis of y through a distance equal to 8 of the units. It is evident, by the solution of the equation $2x^2+x+2=0$, that the two values of x which render the polynomial zero are in this case imaginary. Whenever the number of points in which the curve cuts the axis of x falls short of the degree of the polynomial, it is customary to speak of the curve as *cutting the line in imaginary points.*

11. Maximum and Minimum Values of Polynomials. It is apparent from the considerations established in the preceding Articles, that as the variable x changes from $-\infty$ to $+\infty$, the function $f(x)$ may undergo many variations. It may go on for a certain period increasing, and then, ceasing to increase, may commence to diminish ; it may then cease to diminish and commence again to increase ; after which another period of diminution may arrive, or the function may (as in the last example of the preceding Art.) go on then continually increasing. At a stage where the function ceases to increase and commences to diminish, it is said to have attained a *maximum* value ; and when it ceases to diminish and commences to

increase, it is said to have attained a *minimum* value. A polynomial may have several such values; the number depending in general on the degree of the function. Nothing exhibits so well as a graphic representation the occurrence of such a maximum or minimum value; as well as the various fluctuations of which the values of a polynomial are susceptible.

A knowledge of the maximum and minimum values of a function, giving the positions of the points where the curve bends with reference to the axis is often of great assistance in tracing the curve corresponding to a given polynomial. It will be shown in a subsequent chapter that the determination of these points depends on the solution of an equation one degree lower than that of the given function.

It is easy to show that maxima and minima occur alternately; for, as the variable increases from a value corresponding to one maximum to the value corresponding to a second, the function begins by diminishing and ends by increasing, and, therefore, attains a minimum at some intermediate stage. In like manner it appears that between two minima one maximum must exist.

CHAPTER II

GENERAL PROPERTIES OF EQUATIONS

12. The process of tracing the function $f(x)$ explained in Art. 10 may be employed for the purpose of ascertaining approximately the real roots of a given numerical equation ; for when the corresponding curve is accurately traced, the real roots of the equation $f(x)=0$ can be obtained approximately by measuring the distances from the origin of its points of intersection with the axis. With a view to the more accurate numerical solution of this problem, as well as the general discussion of equations of both numerical and algebraical, we proceed to establish in the present chapter the most important general properties of equations having reference to the existence and number of the roots, and the distinction between real and imaginary roots.

By the aid of the following theorem the existence of a real root in an equation may often be established :—

Theorem. *If two real quantities a and b be substituted for the unknown quantity x in any polynomial $f(x)$, and if they furnish results having different signs, one plus and the other minus ; then the equation $f(x)=0$ must have at least one real root intermediate in value between a and b.*

This theorem is an immediate consequence of the property of the continuity of the function $f(x)$ established in Art. 7 ; for since $f(x)$ changes continuously from $f(a)$ to $f(b)$, and, therefore, passes through all of the intermediate values, while x changes from a to b ; and since one of these quantities, $f(a)$ or $f(b)$, is positive, and the other negative, it follows that for some value of x intermediate between a and b, $f(x)$ must attain the value zero which is intermediate between $f(a)$ and $f(b)$.

The student will assist his conception of this theorem by reference to the graphic method of representation. What is here proved, and what will appear obvious from the figure, is, that if there exist two points of the curved line representing the polynomial on opposite sides of the axis OX, then the curve joining these points must cut that axis at least once. It will also be evident from the figure that several values may exist between a and b for which $f(x)=0$ *i.e.*, for which the curve cuts the axis. For example, in Fig. 3, Art. 10, $x=-2$ gives a negative value (-144), and $x=2$ gives a positive value (20), and

between these points of the curve there exist *three* points of section of the axis of x.

Corollary. *If there exist no real quantity which, substituted for x, makes $f(x)=0$, then $f(x)$ must be positive for every real value of x.*

For it is evident (Art. 4) that $x=\infty$ makes $f(x)$ positive; and no value of x, therefore, can make it negative; for if there were any such value the equation would by the theorem of this Article have a real root, which is contrary to our present hypothesis. With reference to the graphic mode of representation this theorem may be expressed by saying that when the equation $f(x)=0$ has no real root, the curve representing the polynomial $f(x)$ must lie entirely above the axis of x.

13. Theorem. *Every equation of an odd degree has at least one real root of a sign opposite to that of its last term.*

This is an immediate consequence of the theorem in the last Article. Substitute in succession $-\infty$, 0, ∞ for x in the polynomial $f(x)$. The results are, n being odd (see Art. 4),

for $x=-\infty$, $f(x)$ is negative;

„ $x=0$, sign of $f(x)$ is the same as that of a_n,

„ $x=+\infty$, $f(x)$ is positive.

If a_n is positive, the equation must have a real root between $-\infty$ and 0, *i.e.*, a real negative root; and if a_n is negative, the equation must have a root between 0 and ∞, *i.e.*, a real positive root. The theorem is, therefore, proved.

14. Theorem. *Every equation of an even degree, whose last term is negative, has at least two real roots, one positive and the other negative.*

The results of substituting $-\infty$, 0, ∞ are in this case

$$\begin{array}{c} -\infty, +, \\ 0, -, \\ +\infty, +, \end{array}$$

hence there is a real root between $-\infty$ and 0, and another between 0 and $+\infty$ *i.e.*, there exist at least one real negative, and one real positive root.

We have contented ourselves in both this and the preceding Articles with proving the *existence* of roots, and for this purpose it is sufficient to substitute very large positive or negative values, as we have done, for x. It is of course possible to narrow the limits within which the roots lie by the aid of the theorem of Art. 4, and still more by the aid of the theorems respecting the limits of the roots to be given in a subsequent chapter.

15. Existence of a Root in the General Equation. Imaginary Roots. We have now proved the existence of a real root in the case of every equation except one of an even degree whose last

term is positive. Such an equation may have no real root at all. It is necessary then to examine whether, in the absence of real values, there may not be values, involving the imaginary expression $\sqrt{-1}$, which, when substituted for x, reduce the polynomial to zero; or whether there may not be in certain cases both real and imaginary values of the variable which satisfy the equation. We take a simple example to illustrate the occurrence of such imaginary roots. As already remarked (Art. 10), the curve corresponding to the polynomial

$$f(x) \equiv 2x^2 + x + 2$$

lies entirely above the axis of x, as in Fig. 4. The equation $f(x)=0$ has no real roots; but it has the two imaginary roots

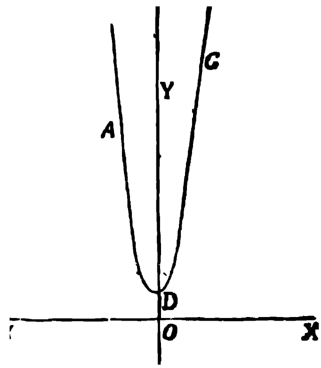

Fig. 4.

$$-\frac{1}{4}+\frac{\sqrt{15}}{4}\sqrt{-1}, \quad -\frac{1}{4}-\frac{\sqrt{15}}{4}\sqrt{-1},$$

as is evident by the solution of the quadratic. We observe, therefore, that in the absence of any real values there are in this case two imaginary expressions which reduce the polynomial to zero.

The corresponding general proposition is, that *every rational integral equation has a root of the form*

$$\alpha+\beta\sqrt{-1},$$

α and β being real finite quantities. This statement includes both real and imaginary roots, the former corresponding to the value $\beta=0$. When α and β are numbers, such an expression is called a *complex number*; and what is asserted is that every numerical equation has a numerical root either real or complex.

As the proof of this proposition involves principles which could not conveniently have been introduced hitherto, and which will present themselves more naturally for discussion in subsequent parts of the work, we defer the demonstration until these principles have been established. For the present, therefore, we assume the proposition, and proceed to derive certain consequences from it.

16. Theorem. *Every equation of n dimensions has n roots, and no more.*

We first observe that if any quantity h is a root of the equation $f(x)=0$, then $f(x)$ is divisible by $x-h$ without a remainder. This is

evident from Art. 9 ; for if $f(h)=0$, *i.e.*, if h is a root of $f(x)=0$, R must be $=0$.

Let now, the given equation be

$$f(x) \equiv x^n + p_1 x^{n-1} + p_2{}^{n-2} + \ldots + p_{n-1}x + p_n = 0.$$

This equation must have a root real, or imaginary (Art. 15), which we shall denote by the symbol α_1. Let the quotient, when $f(x)$ is divided by $x-\alpha_1$, be $\phi_1(x)$; we have then the identical equation

$$f(x) \equiv (x-\alpha_1)\phi_1(x).$$

Again, the equation $\phi_1(x)=0$, which is of $n-1$ dimensions, must have a root, which we represent by α_2. Let the quotient obtained by dividing $\phi_1(x)$ by $x-\alpha_2$ be $\phi_2(x)$. Hence

$$\phi_1(x) \equiv (x-\alpha_2)\phi_2(x),$$

and $\therefore$ $$f(x) \equiv (x-\alpha_1)(x-\alpha_2)\phi_2(x),$$

where $\phi_2(x)$ is of $n-2$ dimensions.

Proceeding in this manner, we prove that $f(x)$ consists of the product of n factors, each containing x in the first degree, and a numerical factor $\phi_n(x)$. Comparing the co-efficients of x^n, it is plain that $\phi_n(x)=1$. Thus we prove the identical equation

$$f(x) \equiv (x-\alpha_1)(x-\alpha_2(x-\alpha_3)\ldots\ldots(x-\alpha_{n-1})(x-\alpha_n)$$

It is evident that the substitution of any one of the quantities $\alpha_1, \alpha_2, \ldots\ldots\ldots, \alpha_n$ for x in the right-hand member of this equation will reduce that member to zero, and will, therefore, reduce $f(x)$ to zero ; that is to say, the equation $f(x)=0$ has for roots the n quantities $\alpha_1, \alpha_2, \alpha_3, \ldots\ldots, \alpha_{n-1}, \alpha_n$. And it can have no other roots ; for if any quantity other than one of the quantities $\alpha_1, \alpha_2, \ldots\ldots, \alpha_n$ be substituted in the right-hand member of the above equation, the factors will be all different from zero, and, therefore, the product cannot vanish.

Corollary. *Two polynomials each of the nth degree in x cannot be equal to one another for more than n values of x without being completely identical.*

For if their difference be equated to zero, we obtain an equation of the nth degree, which can be satisfied by n values only of x, unless each co-efficient be separately equal to zero.

The theorem of this Article, although of no assistance in the solution of the equation $f(x)=0$, enables us to solve completely the converse problems, *i.e.*, to find the equation whose roots are any n given quantities. The required equation is obtained by multiplying together the n simple factors formed by subtracting from x each of the given roots. By the aid of the present theorem also, when any (one or more) of the roots of a given equation are known, the equation containing the remaining roots may be obtained. For this

purpose it is only necessary to divide the given equation by the product of the given binomial factors. The quotient will be the required polynomial composed of the remaining factors.

Examples

1. Find the equation whose roots are

$-3, -1, 4, 5.$ [*Ans.* $x^4-5x^3-13x^2+53x+60=0.$

2. The equation

$$x^4-6x^3+8x^2-17x+10=0$$

has a root 5; find the equation containing the remaining roots.

Use the method of division of Art. 8. [*Ans.* $x^3-x^2+3x-2=0.$

3. Solve the equation

$$x^4-16x^3+86x^2-176x+105=0,$$

two roots being 1 and 7. [*Ans.* The other two roots are 3, 5.

4. Form the equation whose roots are

$-\frac{3}{2}, 3, \frac{1}{7}.$ [*Ans.* $14x^3-23x^2-60x+9=0.$

5. Solve the cubic equation

$$x^3-1=0.$$

Here it is evident that $x=1$ satisfies the equation. Divide by $x-1$, and solve the resulting quadratic The two roots are found to be

$$-\frac{1}{2}+\frac{1}{2}\sqrt{-3}, -\frac{1}{2}-\frac{1}{2}\sqrt{-3}.$$

6. Form an equation with rational co-efficients which shall have for a root the irrational expression

$$\sqrt{p}+\sqrt{q}.$$

This expression has four different values according to the different combinations of the radical signs, *viz.*,

$$\sqrt{p}+\sqrt{q}, \ -\sqrt{p}-\sqrt{q}, \qquad \sqrt{p}-\sqrt{q}, \ -\sqrt{p}+\sqrt{q}.$$

The required equation is, therefore,

$$(x-\sqrt{p}-\sqrt{q})(x+\sqrt{p}+\sqrt{q})(x-\sqrt{p}+\sqrt{q})(x+\sqrt{p}-\sqrt{q})=0,$$

or

$$(x^2-p-q-2\sqrt{pq})(x^2-p-q+2\sqrt{pq})=0.$$

or finally,

$$x^4-2(p+q)x^2+(p-q)^2=0.$$

17. Equal Roots. It must be observed that the n factors of which a polynomial $f(x)$ consists need not be all different from one another. The factor $x-\alpha$, for example, may occur in the second, or any higher power not superior to n. In this case the equation $f(x)=0$ is still said to have n roots, two or more being now equal to one another; and the root α is called a multiple root of the equation —double, triple, etc., according to the number of times the factor is repeated.

A reference to the graphic construction in Art. 10 (Fig. 3) will help to explain the occurrence of multiple roots. We see by an inspection of the figure that the two positive roots of the equation $10x^3-17x^2+x+6=0$ are nearly equal, and we may conceive that a slight addition to the absolute term of this polynomial, which is, as already explained, equivalent to a small parallel movement upwards of the whole curve, would have the effect of rendering equal the roots of the equation thus altered. In that case the line OX would no longer cut the curve in two distinct points, but would *touch* it. Now, when a line touches a curve it is properly said to meet the curve, not once, but in *two coincident points*. The student acquainted with the theory of plane curves will have no difficulty in illustrating in a similar manner the occurrence of a triple or higher multiple root.

Equal roots form the connecting link between real and imaginary roots. We have just seen that a small change in the form of a polynomial may convert it from one having real roots into another in which two of the real roots become equal. A further small change may convert it into a form in which the two roots become imaginary. Let us suppose that the above polynomial is further altered by another small addition to the absolute term. We shall then have a graphic representation in which the axis OX cuts the curve in only one real point, *viz.*, that corresponding to the negative root, the two points of section corresponding to the two positive roots having now disappeared.

Consider, for example, the polynomial $10x^3-17x^2+x+28$, which is obtained from that of Ex. 2, Art. 10 by the addition of 22. The student can easily construct the figure ; the point corresponding to A in Fig. 3 will now lie much above the axis of x. Divide by $x+1$, and obtain the trinomial $10x^2-27x+28$ which contains the remaining two roots. They are easily found to be

$$\frac{27}{20}+\frac{\sqrt{391}}{20}\sqrt{-1},\quad \frac{27}{20}-\frac{\sqrt{391}}{20}\sqrt{-1}.$$

We observe in this case, as well as in the example of Art. 15, that when a change of form of the polynomial causes one real root to disappear, a second also disappears at the same time, and the two are replaced by a pair of imaginary roots. The reason of this will be apparent from the proposition of the following Article.

18. Imaginary Roots enter Equations in Pairs. The proposition to be now proved may be stated as follows :—

If an equation $f(x)=0$, whose co-efficients are all real quantities,

have for a root the imaginary expression $\alpha+\beta\sqrt{-1}$, *it must also have for a root the conjugate imaginary expression* $\alpha-\beta\sqrt{-1}$.

We have the following identity :—

$$(x-\alpha-\beta\sqrt{-1})(x-\alpha+\beta\sqrt{-1})\equiv(x-\alpha)^2+\beta^2.$$

Let the polynomial $f(x)$ be divided by the second number of this identity, and, if possible, let there be a remainder $Rx+R'$. We have then the identical equation

$$f(x)\equiv\{x-\alpha)^2+\beta^2\}Q+Rx+R',$$

where Q is the quotient, of $n-2$ dimensions in x. Substitute in this identity $\alpha+\beta\sqrt{-1}$ for x. This, by hypothesis, causes $f(x)$ to vanish. It also causes $(x-a)^2+\beta^2$ to vanish. Hence

$$R(\alpha+\beta\sqrt{-1})+R'=0,$$

from which we obtain the two equations

$$R\alpha+R'=0, \quad R\beta=0,$$

since the real and imaginary parts cannot destroy one another ; hence

$$R=0, \; R'=0.$$

Thus the remainder $Rx+R'$ vanishes ; and, therefore, $f(x)$ is divisible without remainder by the product of the two factors

$$x-\alpha-\beta\sqrt{-1}, \quad x-\alpha+\beta\sqrt{-1}.$$

The equation has, consequently, the root $\alpha-\beta\sqrt{-1}$ as well as the root $\alpha+\beta\sqrt{-1}$.

Thus the total number of imaginary roots in an equation with real co-efficients is always even ; and every polynomial may be regarded as composed of real factors, each pair of imaginary roots producing a real quadratic factor, and each real root producing a real simple factor. The actual resolution of the polynomial into those factors constitutes the complete solution of the equation.

We observed in Art. 17 that equal roots may be considered as the connecting link between real and imaginary roots. This statement may be regarded from another point of view. Suppose a polynomial has the quadratic factor $(x-\alpha)^2+k$, and let its form be altered by means of slight alterations in the value of k. When k is negative, the quadratic factor gives a pair of *real* roots ; when $k=0$, this factor has two *equal* roots, α ; when k is positive, the factor has two *imaginary* roots.

A proof exactly similar to that given above shows that ***surd roots, of the form*** $\alpha\pm\sqrt{\gamma}$, ***enter equations whose co-efficients are rational in pairs.***

Examples

1. Form a rational cubic equation which shall have for roots

$$1,\ 3+2\sqrt{-1}. \qquad [Ans.\ x^3-7x^2+19x-13=0.$$

2. Form a rational equation which shall have for two of its roots

$$1+5\sqrt{-1},\ 5-\sqrt{-1}.$$

$$[Ans.\ x^4-12x^3+72x^2-312x+676=0.$$

3. Solve the equation

$$x^4+2x^3-5x^2+6x+2=0,$$

which has a root

$$-2+\sqrt{3}.$$

[*Ans.* The roots are $-2\pm\sqrt{3},\ 1\pm\sqrt{-1}$.

Solve the equation

$$3x^3-4x^2+x+88=0,$$

one root being $2+\sqrt{-7}$. [*Ans.* The roots are $2\pm\sqrt{-7},\ -\frac{8}{3}$.

19. Descartes' Rule of Signs—Positive Roots. This rule, which enables us, by the mere inspection of a given equation, to assign a superior limit to the number of its positive roots, may be enunciated as follows :—*No equation can have more positive roots than it has changes of sign from + to —, and from — to +, in the terms of its first member.*

We shall content ourselves for the present with the proof which is usually given, and which is rather a verification than a general demonstration of this celebrated theorem of Descrates. It will be subsequently shown that the rule just enunciated, and other similar rules which were discovered by early investigators to the number of the positive, negative, and imaginary roots of equations, are immediate deductions from the more general theorems of Budan and Fourier.

Let the signs of a polynomial taken at random succeed each other in the following order :—

$$+\ +\ -\ +\ -\ -\ -\ +\ +\ -\ +\ -.$$

In this there are in all seven changes of sign, including changes from + to —, and from — to +. It is proposed to show that if this polynomial be multiplied by a binomial whose signs, corresponding to a positive root, are + —, the resulting polynomial will have at least one more change of sign than the original.

We write down only the signs which occur in the operation as follows :—

$$\begin{array}{ccccccccccccc} + & + & - & + & - & - & - & + & + & - & + & - & \\ & - & - & + & - & + & + & + & - & - & + & - & + \\ \hline + & \pm & - & + & - & \mp & \mp & + & \pm & - & + & - & + \end{array}$$

Here, in the third line, the ambiguous sign $\pm$ is placed wherever there are two terms with different signs to be added. We observe in this case, and it will readily appear also for every other arrangement, that the effect of the process is to introduce the ambiguous sign wherever the sign $+$ follows $+$, or $-$ follows $-$, in the original polynomial: The number of variations of sign is never diminished. There is, moreover, always one variation added at the end. This is obvious in the above instance, where the original polynomial terminates with a variation; if it terminates with a continuations of sign, it will equally appear that the corresponding ambiguity in the resulting polynomial must furnish one additional variation either with the preceding or with the superadded sign. Thus, in even the most unfavourable case—that, namely, in which the continuation of sign in the original remain continuations in the resulting polynomial, there is one variation added; and we may conclude in general that the effect of the multiplication of a polynomial by a binomial factor $x-\alpha$ is to introduce at least one additional change of sign.

Suppose now a polynomial formed of the product of the factors corresponding to the negative and imaginary roots of an equation; the effect of multiplying this by each of the factors $x-\alpha$, $x-\beta$, $x-\gamma$, etc., corresponding to the positive roots α, β, γ, etc., is to introduce at least one change of sign for each; so that when the complete product is formed containing all the roots, we conclude that the resulting polynomial has at least as many changes of sign as it has positive roots. This is Descartes' proposition.

20. Descartes' Rule of Signs—Negative Roots. In order to give the most advantageous statement to Descartes' rule in the case of negative roots, we first prove that if $-x$ be substituted for x in the equation $f(x)=0$, the resulting equation will have the same roots as the original except that their signs will be changed. This follows from the identical equation of Art. 16.

$$f(x)=(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)\ldots.(x-\alpha_n)$$

from which we derive

$$f(-x)\equiv(-1)^n(x+\alpha_1)(x+\alpha_2)(x+\alpha_3)\ldots.(x+\alpha_n).$$

From this it is evident that the roots of $f(-x)=0$ are

$$-\alpha_1,\ -\alpha_2,\ -\alpha_3,\ldots\ldots,\ \alpha_n.$$

Hence the negative roots of $f(x)$ are positive roots of $f(-x)$, and we may enunciate Descartes' rule for negative roots as follows:—*No equation can have a greater number of negative roots than there are changes of sign in terms of the polynomial $f(-x)$.*

21. Use of Descartes' Rule in proving the existence of imaginary Roots. It is often possible to detect the existence of imaginary roots in equations by the application of Descartes' rule; for if it should happen that the sum of the greatest possible number of positive roots, added to the greatest possible number of negative roots, is less than the degree of the equation, we are sure of the existence of imaginary roots. Take, for example, the equation

$$x^8+10x^3+x-4=0.$$

The equation, having only one variation, cannot have more than one positive root. Now, changing x into $-x$, we get

$$x^8-10x^3-x-4=0,$$

and since this has only one variation, the original equation cannot have more than one negative root. Hence, in the proposed equation there cannot exist more than two real roots. It has, therefore, at least six imaginary roots. This application of Descartes' rule is available only in the case of incomplete equations; for it is easily seen that the sum of the number of variations is $f(x)$ and $f(-x)$ is exactly equal to the degree of the equation when it is complete.

22. Theorem. *If two numbers a and b, substituted for x in the polynomial $f(x)$, give results with contrary signs, an odd number of real roots of the equation $f(x)=0$ lies between them; and if they give results with the same sign, either no real root or an even number of real roots lies between them.*

This proposition, of which the theorem in Art. 12 is a particular case, contains in the most general form the conclusions which can be drawn as to the roots of an equation from the signs furnished by its first member when two given numbers are substituted for x. We proceed to prove the first part of the proposition; the second part is proved in a precisely similar manner.

Let the following m roots $\alpha_1, \alpha_2, \ldots\ldots, \alpha_m$, and no others, of the equation $f(x)=0$ lie between the quantities a and b, of which, as usual, we take a to be the lesser.

Let $\varphi(x)$ be the quotient when $f(x)$ is divided by the product of the m factors $(x-\alpha_1)(x-\alpha_2)\ldots\ldots(x-\alpha_m)$. We have, then, the identical equation

$$f(x)\equiv(x-\alpha_1)(x-\alpha_2)\ \ldots..(x-\alpha_m)\ \varphi(x).$$

Putting in this successively $x=a$, $x=b$, we obtain

$$f(a)=(a-\alpha_1)(a-\alpha_2)\ldots\ (a-\alpha_m)\ \varphi(a),$$

$$f(b)=(b-\alpha_1)(b-\alpha_2)\ \ldots..(b-\alpha_m)\ \varphi(b).$$

Now $\varphi(a)$ and $\varphi(b)$ have the same sign; for if they had different signs there would be, by Art. 12, one root at least of the

equation $\varphi(x)=0$ between them. By hypothesis, $f(a)$ and $f(b)$ have different signs ; hence the signs of the products

$$(a-\alpha_1)(a-\alpha_2)\ldots.(a-\alpha_m),$$
$$(b-\alpha_1)(b-\alpha_2)\ldots.(b-\alpha_m),$$

are different ; but the sign of the second is positive since all its factors are positive ; hence the sign of the first is negative ; but all the factors of the first are negative ; therefore their number must be odd, which proves the proposition.

In this proposition it is to be understood that multiple roots are counted a number of times equal to the degree of their multiplicity.

It is instructive to apply the graphic method of treatment to the theorem of the present Article. From this point of view it appears almost intuitively true ; for it is evident that when any two points are connected by a curve, the portion of the curve between these points must cut the axis an odd number of times when the points are on opposite sides of the axis ; and an even number of times or not at all, when the points are on the same side of the axis.

Examples

1. If the signs of the terms of an equation be all positive, it cannot have a positive root.

2. If the signs of the terms of any complete equation be alternately positive and negative, it cannot have a negative root.

3. If an equation consists of a number of terms connected by + signs followed by a number of terms connected by − signs, it has one positive root and no more.

Apply Art. 12, substituting 0 and ∞ ; and Art. 19.

4. If an equation involves only even powers of x, and if all the coefficients have positive signs, it cannot have a real root.

Apply Arts. 19 and 20.

5. If an equation involve only odd powers of x, and if the co-efficients have all positive signs, it has the root zero and no other real root.

6. If an equation be complete, the number of continuations of sign in $f(x)$ is the sume as the number of variations of sign in $f(-x)$.

7. When an equation is complete ; if all its roots be real, the number of positive roots is equal to the number of variations, and the number of negative roots is equal to the number of continuations of sign.

8. An equation having an even number of variations of sign must have its last sign positive, and one having an odd number of variations must have its last sign negative.

Take the highest power of x with positive co-efficient (*see* Art. 4).

9. Hence prove that if an equation has an even number of variations it must have an equal or less even number of positive roots ; and if it has an odd number of variations it must have an equal or less odd number of positive roots ;

in other words, the number of positive roots when less than the number of variations must differ from it by an even number.

Substitute 0 and ∞, apply Art. 22.

10. Find an inferior limit to the number of imaginary roots of the equation

$$x^6 - 3x^2 - x + 1 = 0.$$

[*Ans*. At least two imaginary roots.

11. Find the nature of the roots of the equation

$$x^4 + 15x^2 + 7x - 11 = 0.$$

Apply Arts. 14, 19, 20. [*Ans*. One positive, one negative, two imaginary.

12. Show that the equation

$$x^3 + qx + r = 0,$$

where q and r are essentially positive, has one negative and two imaginary roots.

13. Show that the equation

$$x^3 - qx + r = 0,$$

where q and r are essentially positive, has one negative root; and that the other two roots are either imaginary or both positive.

14. Show that the equation

$$\frac{A^2}{x-a} + \frac{B^2}{x-b} + \frac{C^2}{x-c} + \dots\dots\dots + \frac{L^2}{x-l} = x - m,$$

where $a, b, c, \dots\dots, l$ are numbers all different from one another, cannot have an imaginary root.

Substitute $\alpha + \beta\sqrt{-1}$ and $\alpha - \beta\sqrt{-1}$ in succession for x, and subtract. We get an expression which can vanish only on the supposition $\beta = 0$.

15. Show that the equation

$$x^n - 1 = 0$$

has, when n is even, two real roots, 1 and -1 and no other real root; and, when n is odd, the real root 1, and no other real root.

This and the next example follow readily from Arts. 19 and 20.

16. Show that the equation

$$x^n + 1 = 0$$

has, when n is even, no real root; and, when n is odd, the real root -1, and no other real root.

17. Solve the equation

$$x^4 + 2qx^3 + 3q^2x^2 + 2q^3x - r^4 = 0.$$

This is equivalent to

$$(x^2 + qx + q^2)^2 - q^4 - r^4 = 0.$$

$$\left[\quad Ans. \quad -\frac{1}{2}q + \sqrt{-\frac{3}{4}q^2 + \sqrt{q^4 + r^4}} \right.$$

The different signs of the radicals give four combinations, and the expression here written involves the four roots.

18. Form the equation which has for roots the different values of the expression

$$2 + \theta\sqrt{7} + \sqrt{11 + \theta\sqrt{7}},$$

where $\theta^2 = 1$.

If no restriction had been made by the introduction of θ, this expression would have 8 values. The $\sqrt{7}$ must now be taken with the same sign where it occurs under the second radical and free from it. There are, therefore, only four values in all. [*Ans*. $x^4 - 8x^3 - 12x^2 + 84x - 63 = 0$.

19. Form an equation which has for roots the four values of

$$-9+\theta\sqrt{137}+3\sqrt{34-2\theta\sqrt{137}},$$

where $\theta^2=1$. [*Ans.* $x^4+36x^3-400x^2-3168x+7744=0$.

20. Form an equation with rational co-efficients which shall have for roots all the values of the expression

$$\theta_1\sqrt{p}+\theta_2\sqrt{q}+\theta_3\sqrt{r}.$$

where $$\theta_1^2=1,\ \theta_2^2=1,\ \theta_3^2=1.$$

There are eight different values of this expression, *viz.*,

$$\begin{array}{ll} \sqrt{p}+\sqrt{q}+\sqrt{r}, & -\sqrt{p}-\sqrt{q}-\sqrt{r}, \\ \sqrt{p}-\sqrt{q}-\sqrt{r}, & -\sqrt{p}+\sqrt{q}+\sqrt{r}, \\ -\sqrt{p}+\sqrt{q}-\sqrt{r}, & \sqrt{p}-\sqrt{q}+\sqrt{r}, \\ -\sqrt{p}-\sqrt{q}+\sqrt{r}, & \sqrt{p}+\sqrt{q}-\sqrt{r}. \end{array}$$

Assume

$$x=\theta_1\sqrt{p}+\theta_2\sqrt{q}+\theta_3\sqrt{r}.$$

Squaring, we have

$$x^2=p+q+r+2(\theta_2\theta_3\sqrt{qr}+\theta_3\theta_1\sqrt{rp}+\theta_1\theta_2\sqrt{pq}).$$

Transposing, and squaring again,

$$(x^2-p-q-r)^2=4(qr+rp+pq)+8\theta_1\theta_2\theta_3\sqrt{pqr}\,(\theta_1\sqrt{p}+\theta_2\sqrt{q}+\theta_3\sqrt{r}).$$

Transposing, substituting x for $\theta_1\sqrt{p}+\theta_2\sqrt{q}+\theta_3\sqrt{r}$, and squaring, we obtain the final equation free from radicals

$$[x^4-2x^2(p+q+r)+p^2+q^2+r^2-2qr-2rp-2pq]^2=64pqrx^2.$$

This is an equation of the eighth degree whose roots are the values above written. Since $\theta_1, \theta_2, \theta_3$ have disappeared it is indifferent which of the eight roots $\pm\sqrt{p}\pm\sqrt{q}\pm\sqrt{r}$ is assumed equal to x in the first instance. The final equation is that which would have been obtained if each of 8 roots had been subtracted from x, and the continued product formed, as in Ex. 6, Art. 16.

CHAPTER III

RELATIONS BETWEEN THE ROOTS AND CO-EFFICIENTS OF EQUATIONS, WITH APPLICATIONS TO SYMMETRIC FUNCTIONS OF THE ROOTS

23. Relations between Roots and Co-efficients. Taking for simplicity the co-efficient of the highest power of x as unity, and representing, as in Art. 16, the n roots of an equation by $\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_n$, we have the following identity :—

$$x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots\ldots + p_{n-1}x + p_n \\ \equiv (x-\alpha_1)(x-\alpha_2)(x-\alpha_3)\ldots(x-\alpha_n). \qquad \ldots (1)$$

When the factors of the second member of this identity are multiplied together, the highest power of x in the product is x^n ; the co-efficient of x^{n-1} is the sum of the n quantities $-\alpha_1$, $-\alpha_2$, etc., *viz.*, the roots with their signs changed ; the co-efficient of x^{n-2} is the sum of the products of these quantities taken two by two ; the co-efficient of x^{n-3} is the sum of their products taken three by three ; and so on, the last term being the product of all the roots with their signs changed. Equating, therefore, the co-efficients of x on each side of the identity (1), we have the following series of equations :—

$$\left.\begin{aligned} p_1 &= -(\alpha_1 + \alpha_2 + \alpha_3 + \ldots\ldots + \alpha_n), \\ p_2 &= (\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3 + \ldots\ldots + \alpha_{n-1}\alpha_n), \\ p_3 &= -(\alpha_1\alpha_2\alpha_3 + \alpha_1\alpha_3\alpha_4 + \ldots\ldots + \alpha_{n-2}\alpha_{n-1}\alpha_n), \\ & \\ p_n &= (-1)^n \alpha_1\alpha_2\alpha_3 \ldots\ldots \alpha_{n-1}\alpha_n, \end{aligned}\right\} \quad \ldots(2)$$

which enable us to state the relations between the roots and co-efficients as follows : —

Theorem. *In every algebraic equation, the co-efficient of whose highest term is unity, the co-efficient p_1 of the second term with its sign changed is equal to the sum of the roots.*

The co-efficient p_2 of the third term is equal to the sum of the products of the roots taken two by two.

The co-efficient p_3 of the fourth term with its sign changed is equal to the sum of the products of the roots taken three by three ; and so on, the signs of the co-efficients being taken alternately negative and positive, and the number of roots multiplied ogether in each term of the correponding function of the roots increasing by unity, till finally that function is reached which consists of the product of the n roots.

When the co-efficient a_0 of x^n is not unity (*see* Art. 1), we must divide each term of the equation by it. The sum of the roots is then equal to $-\frac{a_1}{a_0}$; the sum of their products in pairs is equal to $\frac{a_2}{a_0}$; and so on.

Cor. 1. Every root of an equation is a divisor of the absolute term of the equation.

Cor. 2. If the roots of an equation be all positive, the co-efficients (including that of the highest power of x) will be alternately positive and negative; and if the roots be all negative, the co-efficients will be all positive. This is obvious from the equations (2) [*cf.* Arts. 19 and 20].

24. Applications of the Theorem. Since the equations (2) of the preceding Article supply n distinct relations between the n roots and the co-efficients, it might perhaps be supposed that some advantage is thereby gained in the general solution of the equation. Such, however, is not the case, for suppose it were attempted to determine by means of these equations a root, α_1, of the original equation, this could be effected only by the elimination of the other roots by means of the given equations, and the consequent determination of a final equation of which α_1, is one of the roots. Now, in whatever way this final equation is obtained, it must have for solution not only α_1 but each of the other roots $\alpha_2, \alpha_3, \ldots\ldots\ldots, \alpha_n$; for, since all the roots enter in the same manner in the equations (2) if it had been proposed to determine α_2 (or any other root) by the elimination of the rest, our final equation could differ from that obtained for α_1 only by the substitution of α_2 (or that other root) for α_1. The final equation arrived at, therefore, by the process of elimination must have the n quantities $\alpha_1, \alpha_2, \ldots\ldots\ldots, \alpha_n$ for roots; and cannot consequently, be easier of solution than the given equation. This final equation is, in fact, the original equation itself, with the root we are seeking substituted for x. This we shall show for the particular case of a cubic. The process here employed is general, and may be applied to an equation of any degree. Let α, β, γ be the roots of the equation

$$x^3+p_1x^2+p_2x+p_3=0.$$

We have, by Art. 23,

$$\begin{aligned} p_1 &= -(\alpha+\beta+\gamma), \\ p_2 &= \alpha\beta+\alpha\gamma+\beta\gamma, \\ p_3 &= -\alpha\beta\gamma. \end{aligned}$$

Multiplying the first of these equations by α^2, the second by α, and adding the three, we find

$$p_1\alpha^2+p_2\alpha+p_3=-\alpha^3$$

or

$$\alpha^3+p_1\alpha^2+p_2\alpha+p_3=0,$$

which is the given cubic with α in the place of x.

The student can take as an exercise to prove the same result in the case of an equation of the fourth degree. In the corresponding treatment of the general case the successive equations of Art. 23 are to be multiplied by α^{n-1}, α^{n-2}, α^{n-3}, etc., and added.

Although the equations (2) afford, as we have just seen, no assistance in the general solution of the equation, they are often of use in facilitating the solution of numerical equations when any particular relations among the roots are known to exist. They may also be employed to establish the relations which must obtain among the co-efficients of algebraical equations corresponding to known relations among the roots.

Examples

1. Solve the equation

$$x^3-5x^2-16x+80=0,$$

the sum of two of its roots being equal to zero.

Let the roots be α, β, γ. We have then

$$\alpha+\beta+\gamma=\quad 5.$$
$$\alpha\beta+\alpha\gamma+\beta\gamma=-16.$$
$$\alpha\beta\gamma=-80.$$

Taking $\beta+\gamma=0$, we have, from the first of these, $\alpha=5$; and from either the second or third we obtain $\beta\gamma=-16$. We find for β and γ the values 4 and -4. Thus the three roots are 5, 4, -4.

2. Solve the equation

$$x^3-3x^2+4=0,$$

two of its roots being equal.

Let the three roots be α, α, β. We have

$$2\alpha+\beta=3,$$
$$\alpha^2+2\alpha\beta=0,$$

from which we find $\alpha=2$, and $\beta=-1$. The roots are 2, 2, -1.

3. The equation

$$x^4+4x^3-2x^2-12x+9=0$$

has two pairs of equal roots; find them.

Let the roots be α, α, β, β; we have, therefore,

$$2\alpha+2\beta=-4,$$
$$\alpha^2+\beta^2+4\alpha\beta=-2$$

from which we obtain for α and β the values 1 and -3

4. Solve the equation

$$x^3-9x^2+14x+24=0$$

two of whose roots are in the ratio of 3 to 2.

Let the roots be α, β, γ, with the relation $2\alpha=3\beta$. By elimination o α we easily obtain

$$5\beta+2\gamma=18,$$
$$3\beta^2+5\beta\gamma=28,$$

from which we have the following quadratic for β :—

$$19\beta^2-90\beta+56=0.$$

The roots of this are 4, and $\frac{14}{19}$; the former gives for α and γ the values 6 and -1. The three roots are 6, 4, -1. The student will here ask what is the significance of the value $\frac{14}{19}$ of β ; and the same difficulty may have presented itself in the previous examples. It will be observed that in the examples of this nature we never require all the relations between the roots and co-efficients in order to determine the required unknown quantities. The reason of t is is, that the given condition establishes one or more relations amongst the roots. Whenever the equations employed appear to furnish more than one system of values for the roots, the actual roots are easily determined by the condition that they must satisfy the equation (or equations) between the roots and co-efficients which we have not made use of in determining them. Thus, in the present example, the value $\beta=4$ gives a system satisfying the omitted equation

$$\alpha\beta\gamma=-24 ;$$

while the value $\beta=\frac{14}{19}$ gives a system not satisfying this equation, and is, therefore, to be rejected.

5. Solve the equation

$$x^3-9x^2+23x-16=0$$

whose roots are in arithmetical progression.

Let the roots be $\alpha-\delta$, α, $\alpha+\delta$; we have at once

$$3\alpha=9,$$
$$3\alpha^2-\delta^2=23$$

from which we obtain the three roots 1, 3, 5.

6. Solve the equation

$$x^4+2x^3-21x^2-22x+40=0,$$

whose roots are in arithmetical progression.

Assume for the roots $\alpha-3\delta$, $\alpha-\delta$, $\alpha+\delta$, $\alpha+3\delta$. [*Ans.* -5, -2, 1, 4.

7. Solve the equation

$$27x^3+42x^2-28x-8=0,$$

whose roots are in geometric progression.

Assume for the roots $\alpha\rho$, α, $\frac{\alpha}{\rho}$. From the third of the equations (2), Art. 23, we have $\alpha^3=\frac{8}{27}$ or $\alpha=\frac{2}{3}$. Either of the remaining two equations gives a quadratic for ρ. [*Ans.* -2, $\frac{2}{3}$, $\frac{-2}{9}$.

8. Solve the equation

$$3x^4-40x^3+130x^2-120x+27=0,$$

whose roots are in geometric progression.

Assume for the roots $\frac{\alpha}{\rho^3}$, $\frac{\alpha}{\rho}$, $\alpha\rho$, $\alpha\rho^3$. Employ the second and fourth of the equations (2), Art. 23. [*Ans.* $\frac{1}{3}$, 1, 3, 9.

9. Solve the equation

$$x^4+15x^3+70x^2+120x+64=0.$$

whose roots are in geometric progression. [*Ans.* $-1, -2, -4, -8$.

10. Solve the equation

$$6x^3-11x^2+6x-1=0,$$

whose roots are in harmonic progression.

Take the roots to be α, β, γ. We have here tue relation

$$\frac{1}{\alpha}+\frac{1}{\gamma}=\frac{2}{\beta};$$

hence

$$\beta\gamma+\gamma\alpha+\alpha\beta=3\gamma\alpha; \text{ etc.}$$ [*Ans.* $1, \frac{1}{2}, \frac{1}{3}$.

11. Solve the equation

$$81x^3-18x^2-36x+8=0,$$

whose roots are in harmonic progression.

[*Ans.* $\frac{2}{9}, \frac{2}{3}, -\frac{2}{3}$.

12. If the roots of the equation

$$x^3-px^2+qx-r=0$$

be in harmonic progression, show that the mean root is $\frac{3r}{q}$.

13. The equation

$$x^4-2x^3+4x^2+6x-21=0$$

has two roots equal in magnitude and opposite in sign; determine all the roots.

Take $\alpha+\beta=0$, and employ the first and third of equations (2), Art. 23.

[*Ans.* $\sqrt{3}, \sqrt{-3}, 1\pm\sqrt{-6}$.

14. The equation

$$3x^4-25x^3+50x^2-50x+12=0$$

has two roots whose product is 2 ; find all the roots.

[*Ans.* $6, \frac{1}{3}, 1\pm\sqrt{-1}$.

15. One of the roots of the cubic

$$x^3-px^2+qx-r=0$$

is double another ; show that it may be found from a quadratic equation.

16. Show that all the roots of the equation

$$x^n+p_1x^{n-1}+p_2x^{n-2}+\dots\dots+p_{n-1}x+p_n=0$$

can be obtained when they are in arithmetical progression.

Let the roots be $\alpha, \alpha+\delta, \alpha+2\delta, \dots\dots, \alpha+(n-1)\delta$. The first of equations (2) gives

$$-p_1=n\alpha+[1+2+3+\dots\dots+(n-1)]\delta$$

$$=n\alpha+\frac{n(n-1)}{2}\delta. \qquad \dots(1)$$

Again, since the sum of the squares of any number of quantities is equal to the square of their sum minus twice the sum of their products in pairs, we have the equation

$$p_1^2-2p_2=\alpha^2+(\alpha+\delta)^2+(\alpha+2\delta)^2+\dots\dots$$

$$=n\alpha^2+n(n-1)\alpha\delta+\frac{n(n-1)(2n-1)}{6}\delta^2. \qquad \dots(2)$$

Subtracting the square of (1) from n times the equation (2), we find δ^2 in terms of p_1 and p_2. We can then find α from equation (1). Thus all the roots can be expressed in terms of the co-efficients p_1 and p_2.

17. Find the condition which must be satisfied by the co-efficients of the equation

$$x^3-px^2+px-r=0,$$

when two of its roots α, β are connected by a relation $\alpha+\beta=0$. [*Ans.* $pq-r=0$.

18. Find the condition that the cubic

$$x^3-px^2+qx-r=0$$

should have its roots in geometric progression. [*Ans.* $p^3r-q^3=0$.

19. Find the condition that the same cubic should have its roots in harmonic progression (*see* Ex. 12). [*Ans.* $27r^2-9pqr+2q^3=0$.

20. Find the condition that the equation

$$x^4+px^3+qx^2+rx+s=0,$$

should have two roots connected by the relation $\alpha+\beta=0$; and determine in that case two quadratic equations which shall have for roots (1) α, β; and (2) γ, δ.

[*Ans.* $pqr-p^2s-r^2=0$, (1) $px^2+r=0$, (2) $x^2+px+\dfrac{ps}{r}=0$.

21. Find the condition that the biquadratic of Ex. 20 should have its roots connected by the relation $\beta+\gamma=\alpha+\delta$. [*Ans.* $p^3-4pq+8r=0$.

22. Find the condition that the roots α, β, γ, δ of

$$x^4+px^3+qx^2+rx+s=0$$

should be connected by the relation $\alpha\beta=\gamma\delta$. [*Ans.* $p^2s-r^2=0$.

22. Show that the condition obtained in Ex. 22 is satisfied when the roots of the biquadratic are in geometric progression.

25. Depression of an equation when a relation exists between two of its Roots. The examples given in the preceding Article illustrate the use of the equations connecting the roots and co-efficients in determining the roots in particular cases when known relations exist among them. We shall now show in general, that *if a relation of the form $\beta=\varphi(a)$ exist between two of the roots of an equation $f(x)=0$, the equation may be depressed two dimensions.*

Let $\varphi(x)$ be substituted for x in the identity

$$f(x)\equiv a_0x^n+a_1x^{n-1}+\ldots.+a_n,$$

then
$$f(\varphi(x))\equiv a_0(\varphi(x))^n+a_1(\varphi(x))^{n-1}+\ldots.+a_{n-1}\varphi(x)+a_n.$$

We represent, for convenience, the second member of this identity by $F(x)$. Substituting α for x, we have

$$F(\alpha)\equiv f(\varphi(\alpha))\equiv f(\beta)=0;$$

hence α satisfies the equation $F(x)=0$, and it also satisfies the equation $f(x)=0$; hence the polynomials $f(x)$ and $F(x)$ have a common measure $x-\alpha$; thus α can be determined, and from it $\varphi(\alpha)$ or β, and the given equation can be depressed two dimensions.

Examples

1. The equation

$$x^3-5x^2-4x+20=0$$

has two roots whose difference $=3$; find them.

Here $\beta-\alpha=3$, $\beta=3+\alpha$; substitute $x+3$ for x in the given polynomial $f(x)$; it becomes $x^3+4x^2-7x-10$; the common measure of this and $f(x)$ is $x-2$; from which $\alpha=2$, $\beta=5$; the third root is -2.

2. The equation

$$x^4-5x^3+11x^2-13x+6=0,$$

has two roots connected by the relation $2\beta+3\alpha=7$; find all the roots.

[*Ans.* $1, 2, 1\pm\sqrt{-2}$.

It may be observed here, that when two polynomials $f(x)$ and $F(x)$ have common factors, these factors may be obtained by the ordinary process of finding the common measure. Thus if we know that two given equations have common roots, we can obtain these roots by equating to zero the greatest common measure of the given polynomials.

Examples

1. The equations

$$2x^3+5x^2-6x-9=0,$$
$$3x^3+7x^2-11x-15=0$$

have two common roots, find them. [*Ans.* -1 -3.

2. The equations

$$x^3+px^2+qx+r=0,$$
$$x^3+p'x^2+q'x+r'=0$$

have two common roots ; find the quadratic whose roots are these two, and find also the third root of each.

$$\left[Ans.\ \ x^2+\frac{q-q'}{p-p'}x+\frac{r-r'}{p-p'}=0,\ \ \frac{-r(p-p')}{r-r'},\ \ \frac{-r'(p-p')}{r-r'}.\right.$$

26. The Cube Roots of Unity. Equations of the forms

$$x^n-1=0,\ x^n+1=0,$$

consisting of the highest and absolute terms only, are called *binomial equations.* The roots of the former are called the n n^{th} *roots of unity.* A general discussion of these forms will be given in a subsequent chapter. We confine ourselves at present to the simple case of the binomial cubic, for which certain useful properties of the roots can be easily established. It has been already shown (*see* Ex. 5, Art. 16), that the roots of the cubic

$$x^3-1=0$$

are

$$1,\ -\frac{1}{2}+\frac{1}{2}\sqrt{-3},\ -\frac{1}{2}-\frac{1}{2}\sqrt{-3}.$$

If either of the imaginary roots be represented by ω, the other is easily seen to be ω^2, by actually squaring ; or we may see the same thing as follows :—If ω be a root of the cubic, ω^2 must also be a root ; for, since $\omega^3=1$, we get, by squaring, $\omega^6=1$, or $(\omega^2)^3=1$, thus showing that ω^2 satisfies the cubic $x^3-1=0$. We have then the identity

$$x^3-1\equiv(x-1)(x-\omega)(x-\omega^2).$$

Changing x into $-x$, we get the following identity also :—

$$x^3+1\equiv(x+1)(x+\omega)(x+\omega^2),$$

which furnishes the roots of

$$x^3+1=0.$$

Whenever in any product of quantities involving the imaginary cube roots of unity any power higher than the second presents itself, it can be replaced by ω, or ω^2, or by unity ; for example,

$$\omega^4=\omega^3.\omega=\omega,\ \omega^5=\omega^3.\omega^2=\omega^2,\ \omega^6=\omega^3.\omega^3=1, \text{ etc.}$$

The first or second of equations (2), Art. 23, gives the following property of the imaginary cube roots :—

$$1+\omega+\omega^2=0.$$

By the aid of this equation any expression involving real quantities and the imaginary cube roots can be written in any of the forms

$$P+\omega Q,\ P+\omega^2 Q,\ \omega P+\omega^2 Q.$$

Examples

1. Show that the product

$$(\omega m+\omega^2 n)(\omega^2 m+\omega n)$$

is rational. [*Ans.* m^2-mn+n^2.

2. Prove the following identities :—

$$m^3+n^3\equiv(m+n)(\omega m+\omega^2 n)(\omega^2 m+\omega n),$$
$$m^3-n^3\equiv(m-n)(\omega m-\omega^2 n)(\omega^2 m-\omega n)$$

3. Show that the product

$$(\alpha+\omega\beta+\omega^2\gamma)(\alpha+\omega^2\beta+\omega\gamma),$$

is rational. [*Ans.* $\alpha^2+\beta^2+\gamma^2-\beta\gamma-\gamma\alpha-\alpha\beta$.

4. Prove the identity

$$(\alpha+\beta+\gamma)(\alpha+\omega\beta+\omega^2\gamma)(\alpha+\omega^2\beta+\omega\gamma)\equiv\alpha^3+\beta^3+\gamma^3-3\alpha\beta\gamma.$$

5. Prove the identity

$$(\alpha+\omega\beta+\omega^2\gamma)^3+(\alpha+\omega^2\beta+\omega\gamma)^3\equiv(2\alpha-\beta-\gamma)(2\beta-\gamma-\alpha)(2\gamma-\alpha-\beta).$$

Apply Ex. 2.

6. Prove the identity

$$(\alpha+\omega\beta+\omega^2\gamma)^3-(\alpha+\omega^2\beta+\omega\gamma)^3\equiv-3\sqrt{-3}(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta).$$

Apply Ex. 2, and substitute for $\omega-\omega^2$ its value $\sqrt{-3}$.

7. Prove the identity

$$\alpha'^3+\beta'^3+\gamma'^3-3\alpha'\beta'\gamma' \equiv (\alpha^3+\beta^3+\gamma^3-3\alpha\beta\gamma)^2,$$

where

$$\alpha' \equiv \alpha^2+2\beta\gamma,\ \alpha' \equiv \beta^2+2\gamma\alpha,\ \gamma' \equiv \gamma^2+2\alpha\beta.$$

8. Form the equation whose roots are

$$m+n,\ \omega m+\omega^2 n,\ \omega^2 m+\omega n.$$

[*Ans.* $x^3-3mnx-(m^3+n^3)=0$.

9. Form the equation whose roots are

$$l+m+n,\ l+\omega m+\omega^2 n,\ l+\omega^2 m+\omega n.$$

[*Ans.* $x^3-3lx^2+3(l^2-mn)x-(l^3+m^3+n^3-3lmn)=0$.

It is important to observe that, corresponding to the n n^{th} roots of unity, there are n n^{th} roots of any quantity. The roots of the equation

$$x^n-a=0$$

are the n n^{th} roots of a.

The three cube roots, for example, of a are

$$\sqrt[3]{a},\ \omega\sqrt[3]{a},\ \omega^2\sqrt[3]{a},$$

where $\sqrt[3]{a}$ represents the real cube root according to the ordinary arithmetical interpretation. Each of these values satisfies the cubic equation $x^3-a=0$. It is to be observed that the three cube roots may be obtained by multiplying *any one* of the three above written by $1, \omega, \omega^2$.

In addition, therefore, to the real cube roots there are two imaginary cube roots obtained by multiplying the real cube root by the imaginary cube roots of unity. Thus, besides the ordinary cube root 3, the number 27 has the two imaginary cube roots

$$-\frac{3}{2}+\frac{3}{2}\sqrt{-3},\ -\frac{3}{2}-\frac{3}{2}\sqrt{-3},$$

as the student can easily verify by actual cubing.

10. Form a rational equation which shall have

$$\omega\sqrt[3]{Q+\sqrt{Q^2+P^3}}+\omega^2\sqrt[3]{Q-\sqrt{Q^2+P^3}}$$

for a root; where $\omega^3=1$.

Compare Ex. 8. [*Ans.* $x^3+3Px-2Q=0$.

11. Form an equation with rational co-efficients which shall have

$$\theta_1\sqrt[3]{P}+\theta_2\sqrt[3]{Q}$$

for a root where $\theta_1{}^3=1$, and $\theta_2{}^3=1$.

Cubing both sides of the equation

$$x=\theta_1\sqrt[3]{P}+\theta_2\sqrt[3]{Q},$$

and substituting x for its value on the right-hand side, we get

$$x^3-P-Q=3\theta_1\theta_2\sqrt[3]{PQ}\cdot x$$

Cubing again, we have

$$(x^3-P-Q)^3=27PQx^3:$$

Since θ_1 and θ_2 may each have any one of the values 1, ω, ω^2 the nine roots of this equation are

$$\sqrt[3]{P}+\sqrt[3]{Q},\quad \omega\sqrt[3]{P}+\omega\sqrt[3]{Q},\quad \omega^2\sqrt[3]{P}+\omega^2\sqrt[3]{Q},$$
$$\omega^2\sqrt[3]{P}+\omega^2\sqrt[3]{Q},\quad \omega^2\sqrt[3]{P}+\sqrt[3]{Q},\quad \omega\sqrt[3]{P}+\sqrt[3]{Q},$$
$$\omega^2\sqrt[3]{P}+\omega\sqrt[3]{Q},\quad \sqrt[3]{P}+\omega^2\sqrt[3]{Q},\quad \sqrt[3]{P}+\omega\sqrt[3]{Q}.$$

We see also that, since θ_1 and θ_2 have disappeared from the final equation, it is indifferent which of these nine roots is assumed equal to x in the first instance. The resulting equation is that which would have been obtained by multiplying together the nine factors of the form $x-\sqrt[3]{P}-\sqrt[3]{Q}$ obtained from the nine roots above written.

12. Form separately the three cubic equations whose roots are the groups in three (written in vertical columns in Ex. 11) of the roots of the equation of the preceding example.

We can write these down from Ex. 8, taking first m and n equal to $\sqrt[3]{P}$, $\sqrt[3]{Q}$; then equal to $\omega\sqrt[3]{P}$, $\omega\sqrt[3]{Q}$; and finally equal to $\omega^2\sqrt[3]{P}$, $\omega^2\sqrt[3]{Q}$.

[*Ans.* $x^3-3\sqrt[3]{PQ}x-P-Q=0$,

$x^3-3\omega^2\sqrt[3]{PQ}x-P-Q=0$,

$x^3-3\omega\sqrt[3]{PQ}x-P-Q=0$.

27. Symmetric Functions of the Roots. Symmetric functions of the roots of an equation are those functions in which all the roots are alike involved, so that the expression is unaltered in value when any two of the roots are interchanged. For example, the functions of the roots (the sum, the sum of the products in pairs etc.) with which we were concerned in Art. 23 are of this nature ; for, as the student will readily perceive, if in any of these expressions the root α_1, let us say, be written in every place where α_2 occurs, and α_2 in every place where α_1 occurs, the value of the expression will be unchanged.

The functions discussed in Art. 23 are the simplest symmetric functions of the roots, each root entering in the first degree only in any term of any one of them.

We can, without knowing the values of the roots separately in terms of the coefficients, obtain by means of the equations (2) of Art. 23 the values in terms of the coefficients of an infinite variety of symmetric functions of the roots. It will be shown in a subsequent chapter, when the discussion of this subject is resumed, that any rational symmetric function whatever of the roots can be so expressed. The examples appended to this Article, most of which have reference to the simple cases of the cubic and biquadratic, are sufficient for the present to illustrate the usual elementary methods of obtaining such expressions in terms of the coefficients.

It is usual to represent a symmetric function by the Greek letter Σ attached to one term of it, from which the entire expression

may be written down. Thus, if α, β, γ be the roots of a cubic, $\Sigma\alpha^2\beta^2$ represents the symmetric function

$$\alpha^2\beta^2+\alpha^2\gamma^2+\beta^2\gamma^2,$$

where all possible products in pairs are taken, and all the terms added after each is separately squared. Again, in the same case, $\Sigma\alpha^2\beta$ represents the sum

$$\alpha^2\beta+\alpha^2\gamma+\beta^2\gamma+\beta^2\alpha+\gamma^2\alpha+\gamma^2\beta,$$

where all possible permutations of the roots two by two are taken, and the first root in each term then squared.

As an illustration in the case of a biquadratic we take $\Sigma\alpha^2\beta^2$ whose expanded form is as follows :—

$$\alpha^2\beta^2+\alpha^2\gamma^2+\alpha^2\delta^2+\beta^2\gamma^2+\beta^2\delta^2+\gamma^2\delta^2.$$

By the aid of the various symmetric functions which occur among the following examples the student will acquire a facility in writing out in all similar cases the entire expression when the typical term is given.

Examples

1. Find the value of $\Sigma\alpha^2\beta$ of the roots of the cubic equation

$$x^3+px^2+qx+r=0.$$

Multiplying together the equations

$$\alpha+\beta+\gamma=-p,$$

$$\beta\gamma+\gamma\alpha+\alpha\beta=q.$$

we obtain $$\Sigma\alpha^2\beta+3\alpha\beta\gamma=-pq,$$

hence $$\Sigma\alpha^2\beta=3r-pq.$$

2. Find for the same cubic the value of

$$\alpha^2+\beta^2+\gamma^2.$$ [*Ans.* $\Sigma\alpha^2=p^2-2q$.

3. Find for the same cubic the value of

$$\alpha^3+\beta^3+\gamma^3.$$

Multiplying the values of $\Sigma\alpha$ and $\Sigma\alpha^2$, we obtain

$$\alpha^3+\beta^3+\gamma^3+\Sigma\alpha^2\beta=-p^3+2pq\ ;$$

hence, by Ex. 1,

$$\Sigma\alpha^3=-p^3+3pq-3r.$$

4. Find for the same cubic the value of

$$\beta^2\gamma^2+\gamma^2\alpha^2+\alpha^2\beta^2.$$

We easily obtain

$$\beta^2\gamma^2+\gamma^2\alpha^2+\alpha^2\beta^2+2\alpha\beta\gamma(\alpha+\beta+\gamma)=q^2,$$

from which $$\Sigma\alpha^2\beta^2=q^2-2pr.$$

4. Find for the same cubic the value of

$$(\beta+\gamma)(\gamma+\alpha)(\alpha+\beta).$$

This is equal to

$$2\alpha\beta\gamma+\Sigma\alpha^2\beta.$$ [*Ans.* $r-pq$.

6. Find the value of the symmetric function

$$\alpha^2\beta\gamma+\alpha^2\beta\delta+\alpha^2\gamma\delta+\beta^2\alpha\gamma+\beta^2\alpha\delta+\beta^2\gamma\delta+\gamma^2\alpha\beta+\gamma^2\alpha\delta+\gamma^2\beta\delta+\delta^2\alpha\beta+\delta^2\alpha\gamma+\delta^2\beta\gamma$$

of the roots of the biquadratic equation

$$x^4+px^3+qx^2+rx+s=0.$$

Multiplying together

$$\alpha+\beta+\gamma+\delta=-p,$$

$$\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta=-r,$$

we obtain

$$\Sigma\alpha^2\beta\gamma+4\alpha\beta\gamma\delta=pr;$$

hence

$$\Sigma\alpha^2\beta\gamma=pr-4s.$$

7. Find for the same biquadratic the value of the symmetric function

$$\alpha^2+\beta^2+\gamma^2+\delta^2.$$

Squaring $\Sigma\alpha$, we easily obtain

$$\Sigma\alpha^2=p^2-2q.$$

8. Find for the same biquadratic the value of the symmetric function

$$\alpha^2\beta^2+\alpha^2\gamma^2+\alpha^2\delta^2+\beta^2\gamma^2+\beta^2\delta^2+\gamma^2\delta^2.$$

Squaring the equation

$$\Sigma\alpha\beta=q,$$

we obtain

$$\Sigma\alpha^2\beta^2+2\Sigma\alpha^2\beta\gamma+6\alpha\beta\gamma\delta=q^2;$$

hence, by Ex. 6,

$$\Sigma\alpha^2\beta^2=q^2-2pr+2s.$$

9. Find for the same biquadratic the value of $\Sigma\alpha^3\beta$.

To form this symmetric function we take the two permutations $\alpha\beta$ and $\beta\alpha$ of the letters α, β; these give two terms $\alpha^3\beta$ and $\beta^3\alpha$ of Σ. We have similarly two terms from every other pair of the letters α, β, γ, δ; so that the symmetric function consists of 12 terms in all.

Multiply together the two equations

$$\Sigma\alpha\beta=q, \quad \Sigma\alpha^2=p^2-2q;$$

and observe that

$$\Sigma\alpha^2\Sigma\alpha\beta\equiv\Sigma\alpha^3\beta+\Sigma\alpha^2\beta\gamma.$$

[It is convenient to remark here, that results of the kind expressed by this last equation can be verified by the consideration that the number of terms in both members of the equation must be the same. Thus, in the present instance, since $\Sigma\alpha^2$ contains 4 terms, and $\Sigma\alpha\beta$, 6 terms, their product must contain 24; and these are in fact the 12 terms which form $\Sigma\alpha^3\beta$, together with the 12 which form $\Sigma\alpha^2\beta\gamma$.]

Using the results of previous examples, we have, therefore,

$$\Sigma\alpha^3\beta=p^2q-2q^2-pr+4s.$$

10. Find for the same biquadratic the value of

$$\alpha^4+\beta^4+\gamma^4+\delta^4.$$

Squaring $\Sigma\alpha^2$, and employing results already obtained,

$$\Sigma\alpha^4=p^4-4p^2q+2q^2+4pr-4s.$$

11. Find the value, in terms of the coefficients, of the sum of the squares of the roots of the equation

$$x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots+p_n=0.$$

Squaring $\Sigma\alpha_1$, we easily find

$$p_1^2=\Sigma\alpha_1^2+2\Sigma\alpha_1\alpha_2\,;$$

hence

$$\Sigma\alpha_1^2=p_1^2-2p_2.$$

12. Find the value, in terms of the coefficients, of the sum of the reciprocals of the roots of the equation in the preceding example.

From the second last, and last of the equations of Art. 23, we have

$$\alpha_2\alpha_3\ldots\ldots\alpha_n+\alpha_1\alpha_3\ldots\ldots\alpha_n+\ldots\ldots+\alpha_1\alpha_2\ldots\ldots\alpha_{n-1}=(-1)^{n-1}\,p_{n-1},$$

$$\alpha_1\alpha_2\alpha_3\ldots\ldots\alpha_n=(-1)^n\,p_n\,;$$

dividing the former by the latter, we have

$$\frac{1}{\alpha_1}+\frac{1}{\alpha_2}+\frac{1}{\alpha_3}+\ldots\ldots\ldots+\frac{1}{\alpha_n}=\frac{-p_{n-1}}{p_n},$$

or

$$\Sigma\frac{1}{\alpha_1}=\frac{-p_{n-1}}{p_n}.$$

In a similar manner the sum of the products in pairs, in threes etc., of the reciprocals of the roots can be found by dividing the 3rd last, or 4th last, etc. coefficient by the last.

13. Find for the cubic equation

$$a_0x^3+3a_1x^2+3a_2x+a_3=0$$

the value, in terms of the coefficients, of the following symmetric function of the roots α, β, γ :—

$$(\beta-\gamma)^2+(\gamma-\alpha)^2+(\alpha-\beta)^2.$$

N.B. It will often be found convenient to write, as in the present example, an equation with *binomial coefficients,* that is, numerical coefficients the same as those which occur in the expansion by the binomial theorem, in addition to the literal coefficients a_0, a_1, etc. Here the equation being of the third degree, the successive numerical coefficients are those which occur in the expansion to the third power, *viz.*, 1, 3, 3, 1.

We easily obtain

$$a_0^2\,[\,(\beta-\gamma)^2+(\gamma-\alpha)^2+(\alpha-\beta)^2]=18(a_1^2-a_0a_2).$$

14. Express in terms of the coefficients of the cubic in the preceding example the successive coefficients of the quadratic

$$(x-\alpha)^2(\beta-\gamma)^2+(x-\beta)^2(\gamma-\alpha)^2+(x-\gamma)^2(\alpha-\beta)^2=0,$$

where α, β, γ are the roots of the cubic.

Here, in addition to the symmetric function of the preceding example, we have to calculate also the two following :—

$$\alpha(\beta-\gamma)^2+\beta(\gamma-\alpha)^2+\gamma(\alpha-\beta)^2,$$

$$\alpha^2(\beta-\gamma)^2+\beta^2(\gamma-\alpha)^2+\gamma^2(\alpha-\beta)^2.$$

[*Ans.* $(a_0a_2-a_1^2)x^2+(a_0a_3-a_1a_2)x+(a_1a_3-a_2^2)=0.$

15. Find for the cubic of Example 13 the value in terms of the coefficients of

$$(2\alpha-\beta-\gamma)(2\beta-\gamma-\alpha)(2\gamma-\alpha-\beta).$$

Since

$$2\alpha-\beta-\gamma=3\alpha-(\alpha+\beta+\gamma)=3\alpha+\frac{3a_1}{a_0},$$

the required value is easily obtained by substituting $-\frac{a_1}{a_0}$ for x in the identity

$$a_0x^3+3a_1x^2+3a_2x+a^3\equiv a_0(x-\alpha)(x-\beta)(x-\gamma).$$

[*Ans.* $a_0^3(2\alpha-\beta-\gamma)(2\beta-\gamma-\alpha)(2\gamma-\alpha-\beta)=-27(a_0^2a_3-3a_0a_1a_2+2a_1^3)$.

16. Find, in terms of the coefficients of the biquadratic equation

$$a_0x^4+4a_1x^3+6a_2x^2+4a_3x+a_4=0$$

the value of the following symmetric function of the roots :—

$$(\beta-\gamma)^2(\alpha-\delta)^2+(\gamma-\alpha)^2(\beta-\delta)^2+(\alpha-\beta)^2(\gamma-\delta)^2.$$

Here the equation is written with numerical coefficients corresponding to the expansion of the binomial to the 4th power. The symmetric function in question is easily seen to be identical with

$$2\Sigma\alpha^2\beta^2-2\Sigma\alpha^2\beta\gamma+12\alpha\beta\gamma\delta.$$

Employing the results of examples 6 and 8, we find

$$a_0^2\{(\beta-\gamma)^2(\alpha-\delta)^2+(\gamma-\alpha)^2(\beta-\delta)^2+(\alpha-\beta)^2(\gamma-\delta)^2\}=24(a_0a_4-4a_1a_3+3a_2^2).$$

17. Taking the six products in pairs of the four roots of the equation of Ex. 16, and adding each product, *e.g.*, $\alpha\beta$, to that which contains the remaining two roots, $\gamma\delta$, we have the three sums in pairs

$$\beta\gamma+\alpha\delta,\ \gamma\alpha+\beta\delta,\ \alpha\beta+\gamma\delta;$$

it is required to find the values in terms of the coefficients of the two following symmetric functions of the roots :—

$$(\gamma\alpha+\beta\delta)(\alpha\beta+\gamma\delta)+(\alpha\beta+\gamma\delta)(\beta\gamma+\alpha\delta)+(\beta\gamma+\alpha\delta)(\gamma\alpha+\beta\delta),$$
$$(\beta\gamma+\alpha\delta)(\gamma\alpha+\beta\delta)(\alpha\beta+\gamma\delta).$$

The former of these is the sum of the products in pairs, and the latter the continued product, of the three expressions above given. As these three functions of the roots are important in the theory of the biquadratic, we shall represent them uniformly by the letters λ, μ, ν. We have, therefore, to find expressions in terms of the coefficients for $\mu\nu+\nu\lambda+\lambda\mu$, and $\lambda\mu\nu$.

The former is $\Sigma\alpha^2\beta\gamma$, and is easily expressed as follows (*cf.* Ex. 6) :—

$$a_0^2\Sigma\mu\nu=4(4a_1a_3-a_0a_4).$$

The latter is, when multiplied out, equal to

$$\alpha\beta\gamma\delta(\alpha^2+\beta^2+\gamma^2+\delta^2)+\alpha^2\beta^2\gamma^2\delta^2\left(\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}+\frac{1}{\delta^2}\right),$$

and we obtain after easy calculations the following :—

$$a_0^3\lambda\mu\nu=8(2a_0a_3^2-3a_0a_2a_4+2a_1^2a_4).$$

18. Find in terms of the coefficients of the biquadratic of Ex. 16, the value of the following symmetric function of the roots :—

$$\{(\gamma-\alpha)(\beta-\delta)-(\alpha-\beta)(\gamma-\delta)\}\{(\alpha-\beta)(\gamma-\delta)-(\beta-\gamma)(\alpha-\delta)\}$$
$$\{(\beta-\gamma)(\alpha-\delta)-(\gamma-\alpha)(\beta-\delta)\}.$$

This is also an important symmetric function in the theory of the biquadratic. To prevent any ambiguity in writing this, or corresponding functions in which the differences of the roots of the biquadratic enter, we explain the notation which will be uniformly employed in this work.

Taking in circular order the three roots α, β, γ, we have the three differences $\beta-\gamma$, $\gamma-\alpha$, $\alpha-\beta$; and subtracting δ from each root in turn, we have the three other differences $\alpha-\delta$, $\beta-\delta$, $\gamma-\delta$. We combine these in pairs as follows :—

$$(\beta-\gamma)(\alpha-\delta),\ (\gamma-\alpha)(\beta-\delta),\ (\alpha-\beta)(\gamma-\delta).$$

The symmetric function in question is the product of the differences of these three taken as usual in circular order.

Employing the values of λ, μ, ν, in the preceding example, we have

$$-\mu+\nu\equiv(\beta-\gamma)(\alpha-\delta),\ -\nu+\gamma\equiv(\gamma-\alpha)(\beta-\delta),\ -\lambda+\mu\equiv(\alpha-\beta)(\gamma-\delta).$$

We have, therefore, to find the value of

$$(2\lambda-\mu-\nu)(2\mu-\nu-\lambda)(2\nu-\lambda-\mu),$$

or

$$(3\lambda-\Sigma\alpha\beta)(3\mu-\Sigma\alpha\beta)(3\nu-\Sigma\alpha\beta),$$

in terms of the coefficients of the biquadratic.

Multiplying this out, substituting the value of $\Sigma\alpha\beta$, and attending to the results of Ex. 17, we obtain the required expression as follows :—

$$a_0^3(2\lambda-\mu-\nu)(2\mu-\nu-\lambda)(2\nu-\lambda-\mu)=-432\{a_0a_2a_4+2a_1a_2a_3-a_0a_3^2-a_1^2a_4-a_2^3\}$$

The function of the coefficients here arrived at, as well as those before obtained in Examples 13, 15 and 16, will be found to be of great importance in the theory of the cubic and biquadratic equations.

19. Find, in terms of the coefficients of the biquadratic of Ex. 16, the value of the symmetric function

$$(\alpha-\beta)^2+(\alpha-\gamma)^2+(\alpha-\delta)^2+(\beta-\gamma)^2+(\beta-\delta)^2+(\gamma-\delta)^2.$$

This may be represented briefly by $\Sigma(\alpha-\beta)^2$.

[*Ans.* $a_0^2\Sigma(\alpha-\beta)^2=48(a_1^2-a_0a_2)$

20. Prove the following relation between the roots and coefficients of the biquadratic of Ex. 16 :—

$$a_0^3(\beta+\gamma-\alpha-\delta)(\gamma+\alpha-\beta-\delta)(\alpha+\beta-\gamma-\delta)=32(a_0^2a_3-3a_0a_1a_2+2a_1^3).$$

28. Theorems relating to Symmetric Functions. The following two theorems, with which we close for the present the discussion of this subject, will be found useful in many instances in verifying the results of the calculation of the values of symmetric functions in terms of the coefficients.

(1) *The sum of the exponents of all the roots in any term of any symmetric function of the roots is equal to the sum of the suffixes in each term of the corresponding value in the coefficients.*

The sum of the exponents is of course the same for every term of the symmetric function, and may be called the *degree in all the roots* of that function. The truth of the theorem will be observed in the particular cases of the examples 13, 15, 16, 17 etc., of the last Article ; and that it must be true in general appears from the equations (2) of Art. 23, for the suffix of each coefficient in those equations is equal to the degree in the roots of the corresponding function of the roots ; hence in any product of any powers of the coefficients the sum of the suffixes must be equal to the degree in all the roots of the corresponding function of the roots.

(2) *When an equation is written with binomial coefficients, the expression in terms of the coefficients for any symmetric function of the roots, which is a function of their differences only, is such that the algebraic sum of the numerical factors of all the terms in it is equal to zero.*

The truth of this proposition appears by supposing all the coefficients a_0, a_1, a_2 etc., to become equal to unity in the general equal written with binomial coefficients, *viz.*

$$a_0x^n+na_1x^{n-1}+\frac{n(n-1)}{1.2}a_2x^{n-2}+\ldots+a_n=0\ ;$$

for the equation then becomes $(x+1)^n=0$, *i.e.*, all the roots become equal; hence any function of the differences of the roots must in that case vanish, and, therefore, also the function of the coefficients which is equal to it ; but this consists of the algebraic sum of the numerical factors when in it all the coefficients a_0, a_1, a_2 etc., are made equal to unity. In Exs. 13, 15, 16, 18, 20 of Art. 27 we have instances of this theorem.

Examples

1. Find in terms of p, q, r the value of the symmetric function

$$\frac{\beta^2+\gamma^2}{\beta\gamma}+\frac{\gamma^2+\alpha^2}{\gamma\alpha}+\frac{\alpha^2+\beta^2}{\alpha\beta},$$

where α, β, γ are the roots of the cubic equation

$$x^3+px^2+qx+r=0. \qquad [Ans.\ \frac{pq}{r}-3$$

2. Find for the same equation the value of

$$(\beta+\gamma-\alpha)^3+(\gamma+\alpha-\beta)^3+(\alpha+\beta-\gamma)^3. \qquad [Ans.\ 24r-p^3.$$

3. Calculate the value of $\Sigma\alpha^3\beta^3$ of the roots of the same equation. Here $\Sigma\alpha\beta\Sigma\alpha^2\beta^2=\Sigma\alpha^3\beta^3+\alpha\beta\gamma\Sigma\alpha^2\beta$; hence, etc. [*Ans.* $q^3-3pqr+3r^2$.

4. Find for the same equation the value of the symmetric function

$$(\beta^3-\gamma^3)^2+(\gamma^3-\alpha^3)^2+(\alpha^3-\beta^3)^2.$$

$\Sigma\alpha^6$ is easily obtained by squaring $\Sigma\alpha^3$ (*see* Ex. 3, Art. 27).

[*Ans.* $2p^6-12p^4q+12p^3r+18p^2q^2-18pqr-6q^3$.

5. Find for the same equation the value of

$$\frac{\beta^2+\gamma^2}{\beta+\gamma}+\frac{\gamma^2+\alpha^2}{\gamma+\alpha}+\frac{\alpha^2+\beta^2}{\alpha+\beta}. \qquad [Ans.\ \frac{2p^2q-4pr-2q^2}{r-pq}.$$

6. Find for the same equation the value of

$$\frac{\alpha^2+\beta\gamma}{\beta+\gamma}+\frac{\beta^2+\gamma\alpha}{\gamma+\alpha}+\frac{\gamma^2+\alpha\beta}{\alpha+\beta}. \qquad [Ans.\ \frac{p^4-3p^2q+5pr+q^2}{r-pq}.$$

7. Find for the same equation the value of

$$\frac{2\beta\gamma-\alpha^2}{\beta+\gamma-\alpha}+\frac{2\gamma\alpha-\beta^2}{\gamma+\alpha-\beta}+\frac{2\alpha\beta-\gamma^2}{\alpha+\beta-\gamma}. \qquad [Ans.\ \frac{p^4-2p^2q+14pr-8q^2}{4pq-p^3-8r},$$

8. Find the value of the symmetric function $\Sigma\left(\frac{\alpha-\beta}{\alpha+\beta}\right)^2$ for the cubic equation. [*Ans.* $\frac{3p^2q^2-4p^3r-4q^3-2pqr-9r^2}{(r-pq)^2}$.

9. Calculate in terms of p, q, r, s, the value of $\Sigma\,\frac{\alpha\beta}{\gamma^2}$ for the equation

$$x^4+px^3+qx^2+rx+s=0,$$

Here

$$\Sigma\alpha\beta\Sigma\frac{1}{\alpha^2}=\Sigma\frac{\alpha}{\beta}+\Sigma\frac{\alpha\beta}{\gamma^2};\text{ and }\Sigma\alpha\Sigma\frac{1}{\alpha}=4+\Sigma\frac{\alpha}{\beta}.$$

[*Ans.* $\frac{qr^2-2q^2s-prs+4\cdot\cdot}{s^2}$

10. Find the value of $\Sigma\frac{\alpha}{\beta^2}$ of the roots of the equation

$$x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots\ldots+p_{n-1}x+p_n=0.$$

[*Ans.* $\frac{p_{n-1}p_n-p_1p_{n-1}^2+2p_1p_{n-2}p_n}{p_n^2}$.

11. Find for the biquadratic of Ex. 9 the value of

$$(\beta\gamma-\alpha\delta)(\gamma\alpha-\beta\delta)(\alpha\beta-\gamma\delta).$$

Compare Ex. 22. Art. 24. [*Ans.* r^2-p^2s.

12. Find the value of $\Sigma(a_0\alpha+a_1)^2(\beta-\gamma)^2$ in terms of the coefficients of the cubic equation

$$a_0x^3+3a_1x^2+3a_2x+a_3=0.$$

[*Ans.* $\frac{18}{a_0^2}(a_0a_2-a_1^2)^2$.

13. Find the value of the symmetric function $\Sigma\frac{\alpha_1^2+\alpha_2^2}{\alpha_1\alpha_2}$ of the roots of the equation

$$x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots\ldots+p_{n-1}x+p_n=0.$$

The given function may be written in the form

$$\begin{aligned}&\alpha_1\left\{\frac{1}{\alpha_1}+\frac{1}{\alpha_2}+\ldots\ldots\ldots+\frac{1}{\alpha_n}\right\}-1\\&+\alpha_2\left\{\frac{1}{\alpha_1}+\frac{1}{\alpha_2}+\ldots\ldots\ldots+\frac{1}{\alpha_n}\right\}-1\\&+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\&+\alpha_n\left\{\frac{1}{\alpha_1}+\frac{1}{\alpha_2}+\ldots\ldots\ldots+\frac{1}{\alpha_n}\right\}-1\end{aligned}$$

or $\Sigma\alpha_1\Sigma\frac{1}{\alpha_1}-n$, hence. etc. [*Ans.* $\frac{p_1p_{n-1}}{p_n}-n$.

14. Clear of radicals the equation

$$\sqrt{t-\alpha^2}+\sqrt{t-\beta^2}+\sqrt{t-\gamma^2}=0,$$

and express the coefficients of the resulting equation in t in terms of the coefficients of the cubic of Ex. 1. [*Ans.* $3t^2-2(p^2-2q)t-p^4+4p^2q-8pr=0$.

15. If α, β, γ, δ be the roots of the biquadratic of Ex. 9, prove

$$(\alpha^2+1)(\beta^2+1)(\gamma^2+1)(\delta^2+1)=(1-q+s)^2+(p-r)^2.$$

Substitute in turn each of the roots of the equation $x^2+1=0$ in the identity of Art. 16 and multiply.

16. Prove the following relation between the roots and coefficients of the general equation of the n^{th} degree :—

$$(\alpha_1^2+1)(\alpha_2^2+1)\ldots\ldots(\alpha_n^2+1)=(1-p_2+p_4-\ldots)^2+(p_1-p_3+\ldots)^2.$$

17. Find the numerical value of

$$(\alpha^2+2)(\beta^2+2)(\gamma^2+2)(\delta^2+2),$$

where α, β, γ, δ are the roots of the equation

$$x^4-7x^3+8x^2-5x+10=0.$$

Substitute in turn for x each root of the equation $x^2+2=0$, and multiply.

[*Ans.* 166.

18. If $\alpha, \beta, \gamma, \delta$ be the roots of the equation

$$a_0x^4+4a_1x^3+6a_2x^2+4a_3x+a_4=0,$$

prove

$$a_0^3(\beta+\gamma)(\gamma+\alpha)(\alpha+\beta)(\alpha+\delta)(\beta+\delta)(\gamma+\delta)=16\{6a_1a_2a_3-a_0a_3^2+a_1^2a_4\}.$$

The symmetric function in question is equal to $(\mu+\nu)(\nu+\lambda)(\lambda+\mu)$, or $\Sigma\lambda\Sigma\mu\nu-\lambda\mu\nu$, where λ, μ, ν have the values of Ex. 17, Art. 27.

19. Calculate the value of the symmetric function $\Sigma(\alpha-\beta)^4$ of the roots of the biquadratic equation of Ex. 9. [*Ans.* $3p^4-16p^2q+20q^2+4pr-16s$.

20. Show that when the biquadratic is written with binomial coefficients as in Ex. 18, the value of the symmetric function of the preceding example may be expressed in the following form :—

$$a_0^4\Sigma(\alpha-\beta)^4=16\{48(a_0a_2-a_1^2)^2-a_0^2(a_0a_4-4a_1a_3+3a_2^2)\}.$$

21. The distances on a right line of two pairs of points from a fixed origin on the line are the roots (α, β) and (α', β') of the two quadratic equations

$$ax^2+2bx+c=0,\ a'x^2+2b'x+c'=0\ ;$$

prove that when one pair of the points are the harmonic conjugates of the other pair, the following relation exists :—

$$ac'+a'c-2bb'=0.$$

22. The distances of three points A, B, C on a right line from a fixed origin O on the line are the roots of the equation

$$ax^3+3bx^2+3cx+d=0\ ;$$

find the condition that one of the points A, B, C, should bisect the distance between the other two.

Compare Ex. 15, Art. 27. [*Ans.* $a^2d-3abc+2b^3=0$.

23. Retaining the notation of the preceding question, find the condition that the four points O, A, B, C should form a harmonic division.

[*Ans.* $a^2d-3bcd+2c^3=0$.

This can be derived from the result of Ex. 22 by changing the roots into their reciprocals, or it can be easily calculated independently.

24. If the roots $(\alpha, \beta, \gamma, \delta)$ of the equation

$$ax^4+4bx^3+6cx^2+4dx+e=0,$$

be so related that $\alpha-\delta$, $\beta-\delta$, $\gamma-\delta$ are in harmonic progression, prove the relation among the coefficients

$$ace+2bcd-ad^2-b^2e-c^3=0.$$

Compare Ex. 18, Art. 27.

25. Form the equation whose roots are

$$-\frac{\beta\gamma+\omega\gamma\alpha+\omega^2\alpha\beta}{\alpha+\omega\beta+\omega^2\gamma},\ -\frac{\beta\gamma+\omega^2\gamma\alpha+\omega\alpha\beta}{\alpha+\omega^2\beta+\omega\gamma},$$

where $\omega^3=1$, and α, β, γ are the roots of the cubic

$$ax^3+3bx^2+3cx+d=0.$$

[*Ans.* $(ac-b^2)x^2+(ad-bc)x+(bd-c^2)=0$.

Compare Exs. 13 and 14, Art. 27.

26. Express

$$(2\beta\gamma-\gamma\alpha-\alpha\beta)(2\gamma\alpha-\alpha\beta-\beta\gamma)(2\alpha\beta-\beta\gamma-\gamma\alpha)$$

as the sum of two cubes. [*Ans.* $(\beta\gamma+\omega\gamma\alpha+\omega^2\alpha\beta)^3+(\beta\gamma+\omega^2\gamma\alpha+\omega\alpha\beta)^3$.

Compare Ex. 5, Art. 26.

27. **Express**

$$(x+y+z)^3+(x+\omega y+\omega^2 z)^3+(x+\omega^2 y+\omega z)^3$$

in terms of $x^3+y^3+z^3$ and xyz, where $\omega^3=1$. [*Ans.* $3(x^3+y^3+z^3)+18xyz$.

28. If

$$(x^3+y^3+z^3-3xyz)(x'^3+y'^3+z'^3-3x'y'z')\equiv X^3+Y^3+Z^3-3XYZ,$$

find X, Y, Z, in terms of x, y, z ; x', y', z'.

Apply Example 4, Art. 26.

[*Ans.* $X=xx'+yy'+zz'$, $Y=xy'+yz'+zx'$, $Z=xz'+yx'+zy'$.

29. **Resolve**

$$(\alpha+\beta+\gamma)^3\alpha\beta\gamma-(\beta\gamma+\gamma\alpha+\alpha\beta)^3$$

into three factors, each of the second degree in α, β, γ.

[*Ans.* $(\alpha^2+\beta\gamma)(\beta^2-\gamma\alpha)(\gamma^2-\alpha$

Compare Ex. 18, Art. 24.

30. Resolve into simple factors each of the following expressions :—

(1) $(\beta-\gamma)^2(\beta+\gamma-2\alpha)+(\gamma-\alpha)^2(\gamma+\alpha-2\beta)+(\alpha-\beta)^2(\alpha+\beta-2\gamma)$.

(2) $(\beta-\gamma)(\beta+\gamma-2\alpha)^2+(\gamma-\alpha)(\gamma+\alpha-2\beta)^2+(\alpha-\beta)(\alpha+\beta-2\gamma)^2$.

[*Ans.* (1) $2\alpha-\beta-\gamma)(2\beta-\gamma-\alpha)(2\gamma-\alpha-\beta)$.

(2) $-9(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)$.

31. Find the condition that the cubic equation

$$x^3-px^2+qx-r=0$$

should have a pair of roots of the form $a\pm a\sqrt{-1}$; and show how to determine the roots in that case.

If the real root is b, we easily find, by forming the sum of the squares of the roots, $p^2-2q=b^2$. The required condition is

$$(p^2-2q)(q^2-2pr)-r^2=0.$$

32. Solve the equation

$$x^3-7x^2+20x-24=0,$$

whose roots are of the form indicated in Ex. 31. [*Ans.* Roots 3, and $2\pm 2\sqrt{-1}$.

33. Find the conditions that the biquadratic equation

$$x^4-px^3+qx^2-rx+s=0$$

should have roots of the form $a\pm a\sqrt{-1}$, $b\pm b\sqrt{-1}$. Here there must be two conditions among the coefficients, as there are only two independent quantities involved in the roots. [*Ans.* $p^2-2q=0$; $r^2-2qs=0$.

34. Solve the biquadratic

$$x^4+4x^3+8x^2-120x+900=0,$$

whose roots are of the form in Ex. 33. [*Ans.* $3\pm 3\sqrt{-1}$, $-5\mp 5\sqrt{-1}$.

35. If $\alpha+\beta\sqrt{-1}$ be a root of the equation

$$x^3+qx+r=0,$$

prove that 2α will be a root of the equation

$$x^3+qx-r=0.$$

36. Find the condition that the cubic equation

$$x^3+px^2+qx+r=0$$

should have two roots α, β connected by the relation $\alpha\beta+1=0$.

[*Ans.* $1+q+pr+r^2=0$.

37. Find the condition that the biquadratic

$$x^4+px^3+qx^2+rx+s=0$$

should have two roots connected by the relation $\alpha\beta+1=0$.

The condition arranged according to powers of s is

$$1+q+pr+r^2+(p^2+pr-2q-1)s+(q-1)s^2+s^3=0.$$

38. Find the value of $\Sigma(\alpha_1-\alpha_2)^2\,\alpha_3\alpha_4\ldots\ldots\alpha_n$ of the roots of the equation

$$x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots\ldots+p_n=0.$$

This is readily reducible to Ex. 13. [*Ans.* $(-1)^n\{p_1p_{n-1}-n^2p_n\}$.

39. If the roots of the equation

$$a_0x^n+na_1x^{n-1}+\frac{n(n-1)}{1.2}a_2x^{n-2}+\ldots\ldots+a_n=0$$

be in arithmetical progression, show that they can be obtained from the expression

$$-\frac{a_1}{a_0}\pm\frac{r}{a_0}\sqrt{\frac{3(a_1^2-a_0a_2)}{n+1}}$$

by giving to r all the values 1, 3, 5,......, $n-1$, when n is even; and all the values 0, 2, 4, 6,......., $n-1$, when n is odd.

40. Representing the differences of three quantities α, β, γ by $\alpha_1, \beta_1, \gamma_1$, as follows :—

$$\alpha_1\equiv\beta-\gamma,\ \beta_1\equiv\gamma-\alpha,\ \gamma_1\equiv\alpha-\beta\,;$$

prove the relations

$$\alpha_1^3+\beta_1^3+\gamma_1^3=3\alpha_1\beta_1\gamma_1.$$

$$\alpha_1^4+\beta_1^4+\gamma_1^4=\tfrac{1}{2}\{\alpha_1^2+\beta_1^2+\gamma_1^2\}^2.$$

$$\alpha_1^5+\beta_1^5+\gamma_1^5=\tfrac{5}{2}\{\alpha_1^2+\beta_1^2+\gamma_1^2\}\alpha_1\beta_1\gamma_1.$$

These results can be derived by taking $\alpha_1, \beta_1, \gamma_1$ to be roots of the equation

$$x^3+qx-r=0$$

(where the second term is absent since the sum of the roots$=0$), and calculating the symmetric functions $\Sigma\alpha_1^3$, $\Sigma\alpha_1^4$, $\Sigma\alpha_1^5$ in terms of q and r. The process can be extended to form $\Sigma\alpha_1^6$, $\Sigma\alpha_1^7$, etc. The sums of the successive powers are, therefore, all capable of being expressed in terms of the product $\alpha_1\beta_1\gamma_1$ and the sum of squares $\alpha_1^2+\beta_1^2+\gamma_1^2$; the former being equal to r, and the latter to $-2(\beta_1\gamma_1+\gamma_1\alpha_1+\alpha_1\beta_1)$, or $-2q$. These sums can be calculated readily as follows :— By means of $x^3=r-qx$, and the equations derived from this by squaring, cubing, etc., and multiplying by x or x^2, any power of x, say x^p, can be brought by successive reductions to the form $A+Bx+Cx^2$, where A, B, C are functions of q and r. Substituting $\alpha_1, \beta_1, \gamma_1$, and adding, we find $\Sigma\alpha_1^p=3A-2qC$. The student can take as an exercise to prove in this way $\Sigma\alpha_1^7=7q^2r$, $\Sigma\alpha_1^{11}=11qr(q^3-r^2)$.

CHAPTER IV

TRANSFORMATION OF EQUATIONS

29. Transformation of Equations. We can in many instances, without knowing the values of the roots of an equation in terms of the coefficients, transform it by elementary substitutions, or by the aid of the symmetric functions of the roots, into another equation whose roots shall have certain assigned relations to the roots of the proposed. A transformation of this nature often facilitates the discussion of the equation. We proceed to explain the most important elementary transformations of equations.

30. Roots with Signs Changed. To transform an equation into another whose roots shall be equal to the roots of the given equations with contrary signs, let $\alpha_1, \alpha_2, \alpha_3, .., \alpha_n$ be the roots of the equation

$$x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots\ldots+p_{n-1}x+p_n=0.$$

We have then the identity

$$x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots+p_{n-1}x+p_n\equiv(x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_n);$$

changing x into $-y$, we have, whether n be even or odd,

$$y^n-p_1y^{n-1}+p_2y^{n-2}-\ldots\pm p_{n-1}y\mp p_n\equiv(y+\alpha_1)(y+\alpha_2)\ldots(y+\alpha_n).$$

The polynomial in y equated to zero is, therefore, an equation whose roots are $-\alpha_1, -\alpha_2, \ldots, -\alpha_n$; and to effect the required transformation we have only to *change the signs of every alternate term of the given equation beginning with the second.*

Examples

1. Find the equation whose roots are the roots of

$$x^5+7x^4+7x^3-8x^2+x+1=0$$

with their signs changed. [*Ans.* $x^5-7x^4+7x^3+8x^2+x-1=0$.

2. Change the signs of the roots of the equation

$$x^7+3x^5+x^3-x^2+7x+2=0.$$

[Supply the missing terms with zero coefficients.]

[*Ans.* $x^7+3x^5+x^3+x^2+7x-2=0$.

31. To Multiply the Roots by a Given Quantity. To transform an equation whose roots are $\alpha_1, \alpha_2, .., \alpha_n$ into another whose

roots are $m\alpha_1, m\alpha_2, \ldots, m\alpha_n$, we change x into $\dfrac{y}{m}$ in the identity of the preceding Article. Multiplying by m^n, we have

$$y^n + mp_1y^{n-1} + m^2p_2y^{n-2} + \ldots + m^{n-1}p_{n-1}y + m^np^n$$
$$\equiv (y - m\alpha_1)(y - m\alpha_2)\ldots(y - m\alpha_n).$$

Hence, to multiply the roots of an equation by a given quantity m, *we have only to multiply the successive co-efficients, beginning with the second, by* $m, m^2, m^3, \ldots, m^n$.

The present transformation is useful for the purpose of removing the co-efficient of the first term of an equation when it is not unity ; and generally for removing fractional co-efficients from an equation. If there be a co-efficient a_0 of the first term, we form the equation whose roots are $a_0\alpha_1, a_0\alpha_2, \ldots, a_0\alpha_n$; the transformed equation will be divisible by a_0, and after such division the co-efficient of x^n will be unity.

When there are fractional co-efficients, we can get rid of them by multiplying the roots by a quantity m which is the least common multiple of all the denominators of the fractions. In many cases multiplication by a quantity less than the least common multiple will be sufficient for this purpose, as will appear in the following examples :

Examples

1. Change the equation

$$3x^4 - 4x^3 + 4x^2 - 2x + 1 = 0,$$

into another the co-efficient of whose highest term will be unity.

We multiply the roots by 3. [*Ans.* $x^4 - 4x^3 + 12x^2 - 18x + 27 = 0$.

2. Remove the fractional co-efficients from the equation

$$x^3 - \frac{1}{2}x^2 + \frac{2}{3}x - 1 = 0.$$

Multiply the roots by 6. [*Ans.* $x^3 - 3x^2 + 24x - 216 = 0$.

3. Remove the fractional co-efficients from the equation

$$x^3 - \frac{5}{2}x^2 - \frac{7}{18}x + \frac{1}{108} = 0.$$

By noting the factors which occur in the denominators of these fractions, we observe that a number much smaller than the least common multiple will suffice to remove the fractions. If the required multiplier be m, we write the transformed equation thus : —

$$x^3 - m\frac{5}{2}x^2 - m^2\frac{7}{3^2.2}x + \frac{m^3}{3^3.2^2} = 0 ;$$

it is evident that if m be taken $= 6$, each co-efficient will become integral ; hence we have only to multiply the roots by 6. [*Ans.* $x^3 - 15x^2 - 14x + 2 = 0$.

4. Remove the fractional co-efficients from the equation

$$x^4+\frac{3}{10}x^2+\frac{13}{25}x+\frac{77}{1000}=0.$$

The student must be careful in examples of this kind to supply the missing terms with zero co-efficients. The required multiplier is 10.

[*Ans.* $x^4+30x^2+520x+770=0$.

5. Remove fractional co-efficients from the equation

$$x^4-\frac{5}{6}x^3+\frac{5}{12}x^2-\frac{13}{900}=0.$$

[*Ans.* $x^4-25x^3+375x^2-11700=0$.

32. Reciprocal Roots and Reciprocal Equations. To transform an equation into one whose roots are the reciprocals of the roots of the proposed equation, we change x into $\frac{1}{y}$ in the identity of Art. 30. This substitution gives, after certain easy reductions,

$$\frac{1}{y^n}+\frac{p_1}{y^{n-1}}+\frac{p_2}{y^{n-2}}+\ldots\ldots+\frac{p_{n-1}}{y}+p_n$$

$$\equiv\frac{p_n}{y^n}\left(y-\frac{1}{\alpha_1}\right)\left(y-\frac{1}{\alpha_2}\right)\ldots\ldots\left(y-\frac{1}{\alpha_n}\right)$$

or

$$y^n+\frac{p_{n-1}}{p_n}y^{n-1}+\frac{p_{n-2}}{p_n}y^{n-2}+\ldots+\frac{p_1}{p_n}y+\frac{1}{p_n}$$

$$\equiv\left(y-\frac{1}{\alpha_1}\right)\left(y-\frac{1}{\alpha_2}\right)\ldots\left(y-\frac{1}{\alpha_n}\right)$$

hence, if in the given equation we replace x by $1/y$, and multiply by y^n, the resulting polynomial in y equated to zero will have for roots the reciprocals of $\alpha_1, \alpha_2, \ldots, \alpha_n$.

There is a certain class of equations which remain unaltered when x is changed into its reciprocal. These are called *reciprocal equations*. The conditions which must obtain among the co-efficients of an equation in order that it should be one of this class are, by what has been just proved, plainly the following :—

$$\frac{p_{n-1}}{p_n}=p_1,\ \frac{p_{n-2}}{p_n}=p_2,\ \text{etc.},\ \frac{p_1}{p_n}=p_{n-1},\ \frac{1}{p_n}=p_n.$$

The last of these conditions gives $p_n{}^2=1$, or $p_n=\pm 1$. Reciprocal equations are divided into two classes according as p_n is equal to $+1$, or to -1.

(1) In the first case we have the relations

$$p_{n-1}=p_1,\ p_{n-2}=p_2, \ldots,\ p_1=p_{n-1}\ ;$$

which give rise to the *first class of reciprocal equations, in which the co-efficients of the corresponding terms taken from the beginning and end are equal in magnitude and have the same signs.*

(2) In the second case, when $p_n=-1$, we have

$$p_{n-1}=-p_1,\ p_{n-2}=-p_2, \text{etc.}, \ldots, p_1=-p_{n-1};$$

giving rise to the *second class of reciprocal equations, in which corresponding terms counting from the beginning and end are equal in magnitude but different in sign.* It is to be observed that in this case when the degree of the equation is even, say $n=2m$, one of the conditions becomes $p_m=-p_m$, or $p_m=0$; so that in reciprocal equations of the second class, whose degree is even, the middle term is absent.

If α be a root of reciprocal equation, $1/\alpha$ must also be a root, for it is a root of the transformed equation, and the transformed equation is identical with the proposed; hence the roots of a reciprocal equation occur in pairs, $\alpha, \dfrac{1}{\alpha}$; $\beta, \dfrac{1}{\beta}$, etc. When the degree is odd there must be a root which is its own reciprocal; and it is in fact obvious from the form of the equation that -1, or $+1$ is then a root, according as the equation is of the first or second of the above classes. In either case we can divide off by the known factor ($x+1$ or $x-1$), and what is left is a reciprocal equation of even degree and of the first class. In equations of the second class of even degree x^2-1 is a factor, since the equation may be written in the form

$$x^n-1+p_1x(x^{n-2}-1)+\ldots=0.$$

By dividing by x^2-1, this also is reducible to a reciprocal equation of the first class of even degree. Hence all reciprocal equations may be reduced to *those of the first class whose degree is even,* and this may consequently be regarded as *the standard form of reciprocal equations.*

Examples

1. Find the equation whose roots are the reciprocals of the roots of

$$x^4-3x^3+7x^2+5x-2=0.$$

[*Ans.* $2y^4-5y^3-7y^2+3y-1=0.$

2. Reduce to a reciprocal equation of even degree and of first class

$$x^6+\frac{5}{6}x^5-\frac{22}{3}x^4+\frac{22}{3}x^2-\frac{5}{6}x-1=0.$$

[*Ans.* $x^4+\dfrac{5}{6}x^3-\dfrac{19}{3}x^2+\dfrac{5}{6}x+1=0.$

33. To Increase or Diminish the Roots by a Given Quantity. To effect this transformation we change the variable in the polynomial $f(x)$ by the substitution $x=y+h$; the resulting equation in y will have roots each less or greater by h than the given equation in x, according as h is positive or negative. The resulting equation is (*see* Art. 6)

$$f(h)+f'(h)y+\frac{f''(h)}{1.2}y^2+\frac{f'''(h)}{1.2.3}y^3+\ldots=0.$$

There is a mode of formation of this equation which for practical purposes is much more convenient than the direct calculation of the derived functions, and the substitution in them of the given quantity h. This we proceed to explain. Let the proposed equation be

$$a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\ldots\ldots+a_{n-1}x+a_n=0\ ;$$

and suppose the transformed polynomial in y to be

$$A_0y^n+A_1y^{n-1}+A_2y^{n-2}+\ldots\ldots+A_{n-1}\,y+A_n\ ;$$

since $y=x-h$, this is equivalent to

$$A_0(x-h)^n+A_1(x-h)^{n-1}+\ldots\ .\ +A_{n-1}(x-h)+A_n,$$

which must be identical with the given polynomial. We conclude that if the given polynomial be divided by $(x-h)$, the remainder is A_n, and the quotient

$$A_0(x-h)^{n-1}+A_1(x-h)^{n-2}+\ldots\ldots+A_{n-2}(x-h)+A_{n-1}\ ;$$

if this again be divided by $x-h$, the remainder is A_{n-1}, and the quotient

$$A_0(x-h)^{n-2}+A_1(x-h)^{n-3}+\ldots\ldots+A_{n-2}.$$

Proceeding in this way, we are able by a repetition of arithmetical operations, of the kind explained in Art. 8, to calculate in succession the several co-efficients A_n, A_{n-1}, etc., of the transformed equation ; the last, A_0, being equal to a_0. It will appear in a subsequent chapter that the best practical method of solving numerical equations is only an extension of the process employed in the following examples.

Examples

1. Find the equation whose roots are the roots

$$x^4-5x^3+7x^2-17x+11=0,$$

each diminished by 4.

The calculation is best exhibited as follows :—

1	-5	7	-17	11
	4	-4	12	-20
	-1	3	-5	-9
	4	12	60	
	3	15	55	
	4	28		
	7	43		
	4			
	11			

Here the first division of the given polynomial by $x-4$ gives the remainder -9 $(=A_4)$, and the quotient x^3-x^2+3x-5 (*cf.* Art. 8). Dividing this again by $x-4$, we get the remainder 55 $(=A_3)$, and the quotient $x^2+3x+15$. Dividing again, we get the remainder 43 $(=A_2)$, and quotient $x+7$; and dividing this we get $A_1=11$, and $A_0=1$; hence the required transformed equation is

$$y^4+11y^3+43y^2+55y-9=0$$

2. Find the equation whose roots of

$$x^5+4x^3-x^2+11=0,$$

each diminished by 3.

1	0	4	-1	0	11
	3	9	39	114	342
	3	13	38	114	353
	3	18	93	393	
	6	31	131	507	
	3	27	174		
	9	58	305		
	3	36			
	12	94			
	3				
	15				

The transformed equation is, therefore,

$$y^5+15y^4+94y^3+305y^2+507y+353=0.$$

3. Find the equation whose roots are the roots of

$$4x^5-2x^3+7x-3=0,$$

each increased by 2.

The multiplier in this operation is, of course, -2.

[*Ans.* $4y^5-40y^4+158y^3-308y^2+303y-129=0$.

4. Increase by 7 the roots of the equation

$$3x^4+7x^3-15x^2+x-2=0.$$

[*Ans.* $3y^4-77x^3+720y^2-2876y+4058=0$

5. Diminish by 23 the roots of the equation

$$5x^3-13x^2-12x+7=0.$$

The operation may be conveniently performed by first diminishing the roots by 20 and then diminishing the roots of the transformed equation again by 3. The calculation may be exhibited in two stages, as follows, the broken line marking the conclusion of each stage :—

5	−13	−12	7
	100	1740	34560
	87	1728	**34567**
	100	3740	10122
	187	**5468**	**53689**
	100	906	
	287	6374	
	15	951	
	302	**7325**	
	15		
	317		
	15		
	332		

[*Ans.* $5y^3+332y^2+7325y+53689=0$.

34. Removal of Terms. One of the chief uses of the transformation of the preceding article is to remove a certain specified term from an equation. Such a step often facilitates its solution. Writing the transformed equation in descending powers of y, we have

$$a_0y^n+(na_0h+a_1)y^{n-1}+\left\{\frac{n(n-1)}{1.2}a_0h^2+(n-1)a_1h+a_2\right\}y^{n-2}+\ldots=0.$$

If h be such as to satisfy the equation $na_0h+a_1=0$, the transformed equation will want second term. If h be either of the values which satisfy the equation

$$\frac{n(n-1)}{1.2}a_0h^2+(n-1)a_1h+a_2=0,$$

the transformed equation will want the third term ; the removal of the fourth term will require the solution of a cubic for h ; and so on. To remove the last term we must solve the equation $f(h)=0$, which is the original equation itself.

Examples

1. Transform the equation

$$x^3-6x^2+4x-7=0$$

into one which shall want the second term

$$na_0h+a_1=0 \text{ gives } h=2.$$

Diminish the roots by 2. [*Ans.* $y^3-8y-15=0$.

2. Transform the equation

$$x^4+8x^3+x-5=0,$$

into one which shall want the second term.

Increase the roots by 2.

[*Ans.* $y^4-24y^2+65y-55=0$.

3. Transform the equation

$$x^4-4x^3-18x^2-3x+2=0$$

into one which shall want the third term.

The quadratic for h is

$$6h^2-12h-18=0, \text{ giving } h=3,\ h=-1.$$

Thus there are two ways of effecting the transformation.

Diminishing the roots by 3, we obtain

$$y^4+8y^3-111y-196=0 \qquad \ldots(1)$$

Increasing the roots by 1, we obtain

$$y^4-8y^3+17y-8=0. \qquad \ldots(2)$$

35. Binomial Co-efficients. In many algebraical processes it is found convenient to write the polynomial $f(x)$ in the following form :—

$$a_0x^n+na_1x^{n-1}+\frac{n(n-1)}{1.2}a_2x^{n-2}+\ldots+\frac{n(n-1)}{1.2}a_{n-2}x^2+na_{n-1}x+a_n$$

in which each term is affected, in addition to the literal co-efficient, with the numerical co-efficient of the corresponding term in the expansion of $(x+1)^n$ by the binomial theorem. The student will find examples of equations written in this way on referring to Article 27, Examples 13 and 16. The form is one to which any given polynomial can be at once reduced.

We now adopt the following notation :

$$U_n\equiv a_0x^n+na_1x^{n-1}+\frac{n(n-1)}{1.2}a_2x^{n-2}+\ldots+na_{n-1}x+a_n,$$

thus using U with the suffix n to represent the polynomial of the n^{th} degree written with binomial co-efficients.

We have, therefore, changing n into $n-1$, etc.,

$$U_{n-1}\equiv a_0x^{n-1}+(n-1)a_1x^{n-2}+\ldots+(n-1)a_{n-2}x+a_{n-1},$$

$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

$$U_3\equiv a_0x^3+3a_1x^2+3a_2x+a_3,$$

$$U_2\equiv a_0x^2+2a_1x+a_2,$$

$$U_1\equiv a_0x+a_1,$$

$$U_0\equiv a_0.$$

One advantage of the binomial form is that the derived functions can be immediately written down. The first derived function of U_n is, plainly,

$$n\left\{a_0x^{n-1}+(n-1)a_1x^{n-2}+\frac{(n-1)(n-2)}{1.2}a_2x^{n-3}+\ldots+a_{n-1}\right\};$$

or nU_{n-1}; so that the first derived function of a polynomial represented in this way can be formed by applying to the suffix of U the

rule given in Art. 6 with respect to the exponent of the variable. Thus, for example, the first derived of U_4 is formed by multiplying the function by 4 and diminishing the suffix by unity; it is, therefore, $4U_3$, as the student can easily verify.

We proceed now to prove that the substitution of $y+h$ for x transforms the polynomial U_n, or

$$a_0x^n+na_1x^{n-1}+\frac{n(n-1)}{1.2}a_2x^{n-2}+\ldots+na_{n-1}x+a_n,$$

into

$$A_0y^n+nA_1y^{n-1}+\frac{n(n-1)}{1.2}A_2y^{n-2}+\ldots+nA_{n-1}y+A_n,$$

where

$$A_0, A_1, A_2,\ldots\ldots, A_{n-1}, A_n$$

are the functions which result by substituting h for x in

$$U_0, U_1, U_2,\ldots, U_{n-1}, U_n;$$

i.e., $A_0=a_0$, $A_1=a_0h+a_1$, $A_2=a_0h^2+2a_1h+a_2$, etc.

Representing the derived functions of $f(h)$ by suffixes, as explained in Art. 6, we may write the result of the transformation, *viz.*, $f(y+h)$, in the following form :—

$$f(h)+f_1(h)y+\frac{f_2(h)}{1.2}y^2+\ldots+\frac{f_{n-1}(h)}{1.2\ldots n-1}y^{n-1}+\frac{f_n(h)}{1.2\ldots n}y^n;$$

$f(h)$ is the result of substituting h for x in U_n; it is, therefore, A_n; its first derived $f_1(h)$ is, by the above rule, nA_{n-1}; the first derived of this again is $n(n-1)A_{n-2}$; and so on. Making these substitutions, we have the result above stated, which enables us to write down without any calculation the transformed equation.

Examples

1. Find the result of substituting $y+h$ for x in the polynomial

$$a_0x^3+3a_1x^2+3a_2x+a_3.$$

[*Ans.* $a_0y^3+3(a_0h+a_1)y^2+3(a_0h^2+2a_1h+a_2)y+a_0h^3+3a_1h^2+3a_2h+a_3$.

The student will find it a useful exercise to verify this result by the method of calculation explained in Art. 33, which may often be employed with advantage in the case of algebraical as well as numerical examples.

2. Remove the second term from the equation

$$a_0x^3+3a_1x^2+3a_2x+a_3=0.$$

We must diminish the roots by a quantity h obtained from the equation

$$a_0h+a_1=0, \textit{ i.e., } h=\frac{-a_1}{a_0}.$$

Substituting this value of h in A_2, and A_3, the resulting equation in y is

$$y^3+\frac{3(a_0a_2-a_1^2)}{a_0^2}y+\frac{a_0^2a_3-3a_0a_1a_2+2a_1^3}{a_0^3}=0.$$

3. Find the condition that the second and third terms of the equation $U_n=0$ should be capable of being removed by the same substitution.

Here A_1 and A_2 must vanish for the same value of h; and eliminating h between them we find the required condition. [*Ans.* $a_0a_2-a_1^2=0$.

4. Solve the equation

$$x^3+6x^2+12x-19=0$$

by removing its second term.

The third term is removed by the same substitution, which gives

$$y^3-27=0.$$

The required roots are obtained by subtracting 2 from each root of the latter equation.

5. Find the condition that the second and fourth terms of the equation $U_n=0$ should be capable of being removed by the same transformation.

Here the co-efficients A_1 and A_3 must vanish for the same value of h eliminating h between the equations

$$a_0h+a_1=0,\quad a_0h^3+3a_1h^2+3a_2h+a_3=0,$$

we obtain the required condition

$$a_0^2a_3-3a_0a_1a_2+2a_1^3=0.$$

N.B. When this condition holds among the co-efficients of a biquadratic equation its solution is reducible to that of a quadratic; for when the second term is removed the resulting equation is a quadratic for y^2; and from the values of y those of x can be obtained.

6. Solve the equation

$$x^4+16x^3+72x^2+64x-129=0$$

by removing its second term.

The equation in y is

$$y^4-24y^2-1=0.$$

7. Solve in the same manner the equation

$$x^4+20x^3+143x^2+430x+462=0.$$

[*Ans.* The roots are $-7,\ -3,\ -5\pm\sqrt{3}$.

8. Find the condition that the same transformation should remove the second and fifth terms of the equation $U_n=0$.

[*Ans.* $a_0^3a_4-4a_0^2a_1a_3+6a_0a_1^2a_2-3a_1^4=0$.

36. The Cubic. On account of their peculiar interest, we shall consider in this and the next following Articles the equations of the third and fourth degrees, in connection with the transformation of the preceding article. When $y+h$ is substituted for x in the equation

$$a_0x^3+3a_1x^2+3a_2x+a_3=0, \qquad \ldots(1)$$

we obtain

$$a_0y^3+3A_1y^2+3A_2y+A_3=0,$$

where A_1, A_2, A_3 have the values of Art. 35.

If in the transformed equation the second term be absent,

$$A_1=0, \text{ or } h=-\frac{a_1}{a_0}.$$

Substituting this value of h in A_2 and A_3, we find, as in Ex. 2 Art. 35,

$$a_0A_2=a_0a_2-a_1^2,\ a_0^2A_3=a_0^2a_3-3a_0a_1a_2+2a_1^3;$$

hence the transformed cubic, wanting the second term, is

$$y^3+\frac{3}{a_0^2}(a_0a_2-a_1^2)y+\frac{1}{a_0^3}(a_0^2a_3-3a_0a_1a_2+2a_1^3)=0.$$

The functions of the co-efficients here involved are of such importance in the theory of algebraic equations, that it is customary to represent them by single letters. We accordingly adopt the notation

$$a_0a_2-a_1^2\equiv H,\quad a_0^2a_3-3a_0a_1a_2+2a_1^3\equiv G;$$

and write the transformed equation in the form

$$y^3+\frac{3H}{a_0^2}y+\frac{G}{a_0^3}=0. \qquad \ldots(2)$$

If the roots of this equation be multiplied by a_0 it becomes

$$z^3+3Hz+G=0; \qquad \ldots(3)$$

a form which will be found convenient in the subsequent discussion of the cubic. The variable, z, which occurs in the first member of this equation, is equal to a_0y or a_0x+a_1; the original cubic multiplied by a_0^2 being in fact identical with

$$(a_0x+a_1)^3+3H(a_0x+a_1)+G,$$

as the student can easily verify.

If the roots of the original equation be α, β, γ, those of the transformed equation (2) will be

$$\alpha+\frac{a_1}{a_0},\ \beta+\frac{a_1}{a_0},\ \gamma+\frac{a_1}{a_0};$$

or, since

$$\alpha+\beta+\gamma=-\frac{3a_1}{a_0},$$

they may be written as follows :—

$$\tfrac{1}{3}(2\alpha-\beta-\gamma),\ \tfrac{1}{3}(2\beta-\gamma-\alpha),\ \tfrac{1}{3}(2\gamma-\alpha-\beta).$$

We can write down immediately by the aid of the transformed equation the values of the symmetric functions

$$\Sigma(2\alpha-\beta-\gamma)(2\beta-\gamma-\alpha),\ (2\alpha-\beta-\gamma)(2\beta-\gamma-\alpha)(2\gamma-\alpha-\beta)$$

of the roots of the original cubic. The latter will be found to agree with the value already found in Ex. 15, Art. 27.

We may here make with regard to the general equation an important observation: that any symmetric function of the roots $\alpha, \beta, \gamma, \delta$, etc., which is a function of their *differences* only, can be expressed by the functions of the co-efficients which occur in the transformed equation wanting the second term. This is obvious, since the difference of any two roots α', β' of the transformed equation is equal to the difference of the two corresponding roots α, β of the original equation; and any symmetric function of the roots $\alpha', \beta', \gamma', \delta'$, etc., can be expressed in terms of the co-efficients of the transformed equation. For example, in the case of the cubic, all symmetric functions of the roots which contain the differences only can be expressed as functions of a_0, H, and G. Illustrations of this principle will be found among the examples of Art. 27.

37. The Biquadratic. The transformed equation, wanting the second term, is in this case

$$a_0y^4+6A_2y^2+4A_3y+A_4=0,$$

where A_2 and A_3 have the same values as in the preceding article; and where A_4 is given by the equation

$$a_0^3A_4=a_0^3a_4-4a_0^2a_1a_3+6a_0a_1^2a_2-3a_1^4.$$

The transformed equation is, therefore,

$$y^4+\frac{6}{a_0^2}Hy^2+\frac{4}{a_0^3}Gy+\frac{1}{a_0^4}(a_0^3a_4-4a_0^2a_1a_3+6a_0a_1^2a_2-3a_1^4)=0.$$

We might if we please represent the absolute term of this equation by a symbol like H and G, and have thus three functions of the co-efficients, in terms of which all symmetric functions of the differences of the roots of the biquadratic could be expressed. It is more convenient, however, to regard this term as composed of H and another function of the co-efficients determined in the following manner:—We have plainly the identity

$$a_0^3a_4-4a_0^2a_1a_3+6a_0a_1^2a_2-3a_1^4=a_0^2(a_0a_4-4a_1a_3+3a_2^2)-3(a_0a_2-a_1^2)^2.$$

This involves a_0, H, and another function of the co-efficients, *viz.*,

$$a_0a_4-4a_1a_3+3a_2^2,$$

which is of great importance in the theory of the biquadratic. This function is represented by the letter I, giving

$$a_0^3a_4-4a_0^2a_1a_3+6a_0a_1^2a_2-3a_1^4=a_0^2I-3H^2,$$

The transformed equation may now be written

$$y^4+\frac{6H}{a_0^2}y^2+\frac{4G}{a_0^3}y+\frac{a_0^2I-3H^2}{a_0^4}=0. \qquad (1)$$

We can multiply the roots of this equation, as in the case of the cubic of Art. 36, by a_0; and obtain

$$z^4+6Hz^2+4Gz+a_0^2I-3H^2=0. \qquad \ldots(2)$$

This form will be found convenient in the treatment of the algebraic solution of the biquadratic. The variable is the same as in the case of the cubic, *viz.*, a_0x+a_1; the original quadratic multiplied by a_0^3 being in fact identical with

$$(a_0x+a_1)^4+6H(a_0x+a_1)^2+4G(a_0x+a_1)+a_0^2I-3H^2.$$

Any symmetric function of the roots of the original biquadratic equation which contains their differences only can, therefore, be expressed by a_0, H, G and I

If the roots of the original equation be α, β, γ, δ, those of the transformed (1) will be, as is easily seen,

$$\tfrac{1}{4}(3\alpha-\beta-\gamma-\delta),\ \tfrac{1}{4}(3\beta-\gamma-\delta-\alpha),\ \tfrac{1}{4}(3\gamma-\delta-\alpha-\beta),\ \tfrac{1}{4}(3\delta-\alpha-\beta-\gamma).$$

The sum of these $=0$; the sum of their products in pairs $=\dfrac{6H}{a_0^2}$; the sum of their products in threes $=\dfrac{-4G}{a_0^3}$; and for their continued product we have the equation

$$a_0^4(3\alpha-\beta-\gamma-\delta)(3\beta-\gamma-\delta-\alpha)(3\gamma-\delta-\alpha-\beta)(3\delta-\alpha-\beta-\gamma)$$
$$=256(a_0^2I-3H^2).$$

There is another function of the co-efficients to which we wish now to call attention, as it will be found to be of great importance in the subsequent discussion of the biquadratic. It is the function arrived at in Ex. 18, Art. 27, *viz.*,

$$a_0a_2a_4+2a_1a_2a_3-a_0a_3^2-a_1^2a_4-a_2^3.$$

This is denoted by the letter J. The example referred to shows that it is a function of the differences of the roots. It must, therefore, be capable of being expressed in terms of a_0, H, G, and I. We have, in fact, the identity

$$a_0^3J\equiv a_0^2HI-G^2-4H^3,$$

which the student can easily verify.

Or this relation can be derived as follows :—Whenever a function of the co-efficients a_0, a_1, a_2, etc., is the expression of a

function of the differences of the roots, it must be unaltered by the transformation which removes the second term of the equation; hence its value is unaltered when we change a_1 into zero, a_2 into A_2, a_3 into A_3 etc. Thus

$$a_0a_2a_4+2a_1a_2a_3-a_0a_3^2-a_1^2a_4-a_2^3\equiv a_0A_2A_4-a_0A_3^2-A_2^3;$$

substituting for A_2, A_3, A_4 their values in terms of H, G, I, we easily obtain the above identity, which will usually be written in the form

$$G^2+4H^3\equiv a_0^2(HI-a_0J).$$

38. Homographic Transformation. The transformation of a polynomial considered in Art. 33 is a particular case of the following, in which x is connected with the new variable y by the equation

$$y=\frac{\lambda x+\mu}{\lambda' x+\mu'}.$$

If $\lambda=1$, $\mu=-h$, $\lambda'=0$, $\mu'=1$, we have $y=x-h$, as in Art. 33. Solving for x in terms of y, we have

$$x=\frac{\mu-u'y}{\lambda'y-\lambda}.$$

This value can be substituted for x in the given equation, and the resulting equation of the n^{th} degree in y obtained.

Let α, β, γ, δ, etc., be the roots of the original equation, and α', β', γ', δ', etc., the corresponding roots of the transformed equation. From the equations

$$\alpha'=\frac{\lambda\alpha+\mu}{\lambda'\alpha+\mu'},\ \beta'=\frac{\lambda\beta+\mu}{\lambda'\beta+\mu'},\ \text{etc.},$$

we easily derive the relation

$$\alpha'-\beta'=\frac{(\lambda\mu'-\lambda'\mu)(\alpha-\beta)}{(\lambda'\alpha+\mu')(\lambda'\beta+\mu')},$$

with corresponding relations for the differences of any other pair of roots. If we take any four roots, and the four corresponding roots, we obtain the equation

$$\frac{(\alpha'-\beta')(\gamma'-\delta')}{(\alpha'-\gamma')(\beta'-\delta')}=\frac{(\alpha-\beta)(\gamma-\delta)}{(\alpha-\gamma)(\beta-\delta)}.$$

Thus, if the roots of the proposed equation represent the distances of a number of points on a right line from a fixed origin on the line, the roots of the transformed equation will represent the distances of a corresponding system of points, so related to the former that the anharmonic ratio of any four of one system is the same as that of their four conjugates in the other system. It is in consequence of this property that the transformation is called *homographic*.

It is important to observe that the transformation here considered, in which the variables x and y are connected by a relation of the form

$$Axy+Bx+Cy+D=0,$$

is the most general transformation in which to one value of either variable corresponds one, and only one, value of the other.

39. Transformation by Symmetric Functions. Suppose it is required to transform an equation into another whose roots shall be given rational functions of the roots of the proposed. Let the given function be $\phi(\alpha, \beta, \gamma \ldots)$, where ϕ may involve all the roots, or any number of them. We form all possible combinations $\phi(\alpha, \beta, \gamma)$, $\phi(\alpha, \beta, \delta)$, etc., of the roots of this type, and write down the transformed equation as follows :—

$$[y-\phi(\alpha\ \beta\ \gamma \ldots)][y-\phi(\alpha\ \beta\ \delta \ldots)]\ldots=0.$$

When this product is expanded, the successive co-efficients of y will be symmetric functions of the roots α, β, γ, etc., of the given equation; and may, therefore, be expressed in terms of the coefficients of that equation.

Examples

1. The roots of

$$x^3+px^2+qx+r=0$$

are α, β, γ; find the equation whose roots are α^2, β^2, γ^2.

Suppose the transformed equation to be

$$y^3+Py^2+Qy+R=0;$$

then

$$-P=\alpha^2+\beta^2+\gamma^2, \quad Q=\Sigma\alpha^2\beta^2, \quad -R=\alpha^2\beta^2\gamma^2,$$

and we have to form the symmetric functions $\Sigma\alpha^2$, $\Sigma\alpha^2\beta^2$, $\alpha^2\beta^2\gamma^2$ of the given equation. We easily obtain

$$\Sigma\alpha^2=p^2-2q, \quad \Sigma\alpha^2\beta^2=q^2-2pr, \quad \alpha^2\beta^2\gamma^2=r^2,$$

the transformed equation is, therefore,

$$y^3-(p^2-2q)y^2+(q^2-2pr)y-r^2=0.$$

2. Find in the same case the equation whose roots are α^3, β^3, γ^3.

[*Ans.* $y^3+(p^3-3pq+3r)y^2+(q^3-3pqr+3r^2)y+r^3=0$.

3. If α, β, γ, δ be the roots of

$$x^4+px^3+qx^2+rx+s=0,$$

find the equation whose roots are α^2, β^2, γ^2, δ^2.

Let the transformed equation be

$$y^4+Py^3+Qy^2+Ry+S=0,$$

then

$$-P=\Sigma\alpha^2, \quad Q=\Sigma\alpha^2\beta^2, \quad -R=\Sigma\alpha^2\beta^2\gamma^2, \quad S=\alpha^2\beta^2\gamma^2\delta^2.$$

Compare Exs. 8, 17, Art. 27.

[*Ans.* $y^4-(p^2-2qy)^3+(q^2-2pr+2s)y^2-(r^2-2qs)y+s^2=0$.

4. If α, β, γ, δ be the roots of

$$a_0x^4+4a_1x^3+6a_2x^2+4a_3x+a_4=0;$$

find the equation whose roots are λ, μ, ν; *viz.*,

$$\beta\gamma+\alpha\delta,\ \gamma\alpha+\beta\delta,\ \alpha\beta+\gamma\delta.$$

See Ex. 17, Art. 27.

$$\left[\textit{Ans.}\quad y^3+\frac{6a_2}{a_0}y^2+\frac{4}{a_0^2}(4a_1a_3+a_0a_4)y-\frac{8}{a_0^3}(2a_0a_3^2-3a_0a_2a_4+2a_1^2a_4)=0.\right.$$

5. Show that the transformed equation, when the roots of the resulting cubic of Ex. 4 are multiplied by $\frac{1}{2}a_0$, and the second term of the equation then removed, is

$$z^3-Iz+2J=0.$$

40. Formation of the Equation whose Roots are any Powers of the Roots of the Proposed. The method of effecting this transformation by symmetric functions as explained in the preceding article, is often laborious. A much simpler process, involving multiplication only, can be employed. It depends on a knowledge of the solution of the binomial equation $x^n-1=0$. This form of equation will be discussed in the next chapter. The general process will be sufficiently obvious to the student from the application to the equations of the 2nd and 3rd degrees which will be found among the following examples :—

Examples

1. Form the equation whose roots are the squares of the roots of

$$x^n+p_1x^{n-1}+p_2x^{n-2}+\dots\dots+p_{n-1}x+p_n=0.$$

To effect this transformation, we have the identity

$$x^n+p_1x^{n-1}+p_2x^{n-2}+\dots\dots+p_{n-1}x+p_n\equiv(x-\alpha_1)(x-\alpha_2)\dots\dots(x-\alpha_n);$$

changing x into $-x$, we derive, as in Art. 30,

$$x^n-p_1x^{n-1}+p_2x^{n-2}-\dots\dots\pm p_{n-1}x\mp p_n\equiv(x+\alpha_1)(x+\alpha_2)\ \dots\ (x+\alpha_n);$$

multiplying, we have

$$(x^n+p_2x^{n-2}+p_4x^{n-4}+\dots)^2-(p_1x^{n-1}+p_3x^{n-3}+\dots)^2$$
$$\equiv(x^2-\alpha_1^2)(x^2-\alpha_2^2)\dots(x^2-\alpha_n^2);$$

it is evident that the first member of this identity contains, when expanded, only even powers of x; we may then replace x^2 by y, and obtain finally

$$y^n+(2p_2-p_1^2)y^{n-1}+(p_2^2-2p_1p_3+2p_4)y^{n-2}+\dots\equiv(y-\alpha_1^2)(y-\alpha_2^2)\dots\dots(y-\alpha_n^2).$$

The first member of this equated to zero is the required transformed equation.

N.B. This transformation will often enable us to determine a limit to the number of real roots of the proposed equation. For, a square of a real root must be positive; and, therefore, the original equation cannot have more real roots than the transformed has positive roots.

2. Find the equation whose roots are the squares of the roots of

$$x^3-x^2+8x-6=0. \qquad [\textit{Ans.}\ y^3+15y^2+52y-36=0.$$

The latter equation, by Descartes' rule of signs, cannot have more than one positive root : hence the former must have a pair of imaginary roots.

3. Find the equation whose roots are the squares of the roots of

$$x^5+x^3+x^2+2x+3=0.$$

[*Ans.* $y^5+2y^4+5y^3+3y^2-2y-9=0.$

It follows from Descartes' rule of signs that the original equation must have four imaginary roots.

4. Verify by the method of Ex. 1 the Examples 1 and 3 of Art. 39.

5. Form the equation whose roots are the cubes of the roots of

$$x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots+p_{n-1}x+p_n=0.$$

It will be observed that in Ex. 1 the process consists in multiplying together $f(x)$, the given polynomial, and $f(-x)$; the variables involved in these being those which are obtained by multiplying x by the two roots of the equation $x^2-1=0$. In the present case we must multiply together $f(x)$, $f(\omega x)$, $f(\omega^2 x)$: the variables involved being obtained by multiplying x by the roots of the equation $x^3-1=0$. The transformation may be conveniently represented as follows :

Write the polynomial $f(x)$ in the form

$$(p_n+p_{n-3}x^3+\ldots)+x(p_{n-1}+p_{n-4}x^3+\ldots)+x^2(p_{n-2}+p_{n-5}x_3+\ldots)$$

which we represent, for brevity, by

$$P+xQ+x^2R,$$

where P, Q, and R are all functions of x^3.

We have then

$$P+xQ+x^2R\equiv(x-\alpha_1)(x-\alpha_2)\ldots\ldots(x-\alpha_n) \qquad \ldots(1)$$

Changing, in this identity, x into ωx and $\omega^2 x$ successively, we obtain

$$P+\omega xQ+\omega^2x^2R\equiv(\omega x-\alpha_1)(\omega x-\alpha_2)\ldots(\omega x-\alpha_n), \qquad \ldots(2)$$

$$P+\omega^2xQ+\omega x^2R\equiv(\omega^2x-\alpha_1)(\omega^2x-\alpha_2)\ldots(\omega^2x-\alpha_n), \qquad \ldots(3)$$

since P, Q and R, being functions of x^3, are unaltered.

Multiplying together both members of (1), (2), (3), and attending to the results of Art. 26, we obtain

$$P^3+x^3Q^3+x^6R^3-3x^3PQR\equiv(x^3-\alpha_1^3)(x^3-\alpha_2^3)\ldots(x^3-\alpha_n^3).$$

The first member of this identity contains x in powers which are multiples of 3 only. We can, therefore, substitute y for x^3 and obtain the required transformed equation.

6. Find the equation whose roots are the cubes of the roots of

$$x^4-x^3+2x^2+3x+1=0.$$

[*Ans.* $y^4+14y^3+50y^2+6y+1=0.$

7. Verify by the method of Ex. 5 the result of Ex. 2 of Art. 39.

8. Form the equation whose roots are the cubes of the roots of

$$ax^3+3bx^2+3cx+d=0.$$

[*Ans.* $a^3y^3+3(a^2d+9b^3-9abc)y^2+3(ad^2+9c^3-9bcd)y+d^3=0.$

41. Transformation in General. In the general problem of transformation we have to form a new equation in y, whose roots are connected by a given relation $\varphi(x, y)=0$ with the roots of the

proposed equation $f(x)=0$. The transformed equation will then be obtained by substituting in the given equation the value of x in terms of y derived from the given relation $\varphi(x, y)=0$; or, in other words, by eliminating x between the two equations $f(x)=0$, and $\varphi(x, y)=0$. For example, suppose it were required to form the equation whose roots are the sums of every two of the roots (α, β, γ) of the cubic

$$x^3-px^2+qx-r=0.$$

We have here

$$y=\beta+\gamma=\alpha+\beta+\gamma-\alpha=p-\alpha.$$

The equation $\varphi(x, y)=0$ is in this case $y=p-x$; for when x takes the value α, y takes one of the proposed values; and when x takes the values β and γ, y takes the other proposed values. The transformed equation is, therefore, obtained by substituting $p-y$ for x in the given equation.

Examples

1. If α, β, γ be the roots of the cubic

$$x^3-px^2+qx-r=0,$$

form the equation whose roots are

$$\beta\gamma+\frac{1}{\alpha},\ \gamma\alpha+\frac{1}{\beta},\ \alpha\beta+\frac{1}{\gamma}.$$

Here

$$y=\beta\gamma+\frac{1}{\alpha}=\frac{\alpha\beta\gamma+1}{\alpha}=\frac{1+r}{\alpha};$$

and the given relation is $xy=1+r$; the transformed equation is then obtained by substituting $\dfrac{1+r}{y}$ for x in $f(x)=0$.

[*Ans.* $ry^3-q(1+r)y^2+p(1+r)^2y-(1+r)^3=0$.

2. Form, for the same cubic, the equation whose roots are

$$\alpha\beta+\alpha\gamma,\ \alpha\beta+\beta\gamma,\ \beta\gamma+\alpha\gamma.$$

Substitute $\dfrac{r}{q-y}$ for x. [*Ans.* $y^3-2qy^2+(pr+q^2)y+r^2-pqr=0$.

Form, for the same cubic, the equation whose roots are

$$\frac{\alpha}{\beta+\gamma-\alpha},\ \frac{\beta}{\gamma+\alpha-\beta},\ \frac{\gamma}{\alpha+\beta-\gamma}.$$

Substitute $\dfrac{py}{1+2y}$ for x.

[*Ans.* $(p^3-4pq+8r)y^3+(p^3-4pq+12r)y^2+(6r-pq)y+r=0$.

3. If α, β, γ be the roots of the cubic

$$ax^3+3bx^2+3cx+d=0,$$

prove that the equation in y whose roots are

$$\frac{\beta\gamma-\alpha^2}{\beta+\gamma-2\alpha},\ \frac{\gamma\alpha-\beta^2}{\gamma-\alpha-2\beta},\ \frac{\alpha\beta-\gamma^2}{\alpha+\beta-2\gamma}$$

is obtained by the homographic transformation

$$axy+b(x+y)+c=0.$$

42. Equation of Squared Differences of a Cubic.—We shall now apply the transformation explained in the preceding article to an important problem, *viz.*, the formation of the equation whose roots are the squares of the differences of every two of the roots of a given cubic. We shall do this in the first instance for the cubic

$$x^3+qx+r=0, \qquad \ldots(1)$$

in which the second term is absent, and to which the general equation is readily reducible. Let the roots be α, β, γ. We have to form the equation in y whose roots are

$$(\beta-\gamma)^2,\ (\gamma-\alpha)^2,\ (\alpha-\beta)^2.$$

We may here observe that the method of Art. 39 can be applied to the solution of the general problem, *viz.*, the formation of the equation whose roots are the squares of the differences of every two of the roots of a given equation; for when the product

$$\{y-(\alpha_1-\alpha_2)^2\}\{y-(\alpha_1-\alpha_3)^2\}\{y-(\alpha_1-\alpha_4)^2\}\ldots\{y-(\alpha_2-\alpha_3)^2\}\ldots$$

is formed, the co-efficients of the successive powers of y will be symmetric functions of α_1, α_2, α_3, α_4 etc., and may, therefore, be expressed in terms of the co-efficients of the given equation. In the present instance, however, the method of Art. 41 leads more readily to the required transformed equation. This equation may be called for brevity the "equation of squared differences" of the proposed equation. Assuming y equal to any one of the roots of the transformed equation, *e.g.*, $(\beta-\gamma)^2$, we have

$$y=(\beta-\gamma)^2=\alpha^2+\beta^2+\gamma^2-\alpha^2-\frac{2\alpha\beta\gamma}{\alpha};$$

also

$$\alpha^2+\beta^2+\gamma^2=-2q,\ \alpha\beta\gamma=-r.$$

The equation $\phi(x, y)=0$ of Art. 41 becomes, therefore,

$$y=-2q-x^2+\frac{2r}{x},$$

or

$$x^3+(y+2q)x-2r=0;$$

subtracting from this the proposed equation, we get

$$(y+q)x-3r=0, \text{ or } x=\frac{3r}{y+q};$$

hence the transformed equation in y is

$$y^3+6qy^2+9q^2y+4q^3+27r^2=0. \qquad \ldots(2)$$

If it be proposed to form the equation whose roots are the squares of the differences of the roots (α, β, γ) of the cubic

$$a_0x^3+3a_1x^2+3a_2x+a_3=0, \qquad \ldots(3)$$

we first remove the second term ; the resulting equation is

$$y^3+\frac{3H}{a_0^2}y+\frac{G}{a_0^3}=0;$$

and the required equation is the same as the equation of squared differences of this latter, since the difference of any two roots is unaltered by removing the second term We can, therefore, write down the required equation by putting

$$q=\frac{3H}{a_0^2},\quad r=\frac{G}{a_0^3}$$

in the above. The result is

$$x^3+\frac{18H}{a_0^2}x^2+\frac{81H^2}{a_0^4}x+\frac{27}{a_0^6}(G^2+4H^3)=0, \qquad \ldots(4)$$

which has for roots

$$(\beta-\gamma)^2,\ (\gamma-\alpha)^2,\ (\alpha-\beta)^2.$$

The equation (4) can be written in a form free from fractions by multiplying the roots by a_0^2. It becomes then

$$x^3+18Hx^2+81H^2x+27(G^2+4H^3)=0, \qquad \ldots(5)$$

whose roots are

$$a_0^2(\beta-\gamma)^2,\ a_0^2(\gamma-\alpha)^2,\ a_0^2(\alpha-\beta)^2.$$

We can write down from this an important function of the roots of the cubic (3), *viz.*, *the product of the squares of the differences*, in terms of the co-efficients :—

$$a_0^6(\beta-\gamma)^2(\gamma-\alpha)^2(\alpha-\beta)^2=-27(G^2+4H^3). \qquad \ldots(6)$$

It is evident from the identity of Art. 37 that G^2+4H^3 contains a_0^2 as a factor. We have in fact

$$G^2+4H^3=a_0^2\{a_0^2a_3^2-6a_0a_1a_2a_3+4a_0a_2^3+4a_1^3a_3-3a_1^2a_2^2\}.$$

The expression in brackets is called the ***discriminant*** of the cubic, and is represented by Δ ; giving the identities

$$G^2+4H^3\equiv a_0^2\Delta,\quad HI-a_0J\equiv\Delta.$$

Examples

1. Form the equation of squared differences of the cubic

$$x^3-7x+6=0. \qquad [Ans.\ x^3-42x^2+441x-400=0.$$

2. Form the equation of squared differences of

$$x^3+6x^2+7x+2=0.$$

First remove the second term. [*Ans.* $x^3-30x^2+225x-68=0.$

3. Form the equation of squared differences of

$$x^3+6x^2+9x+4=0. \qquad [Ans.\ x^3-18x^2+81x=0.$$

4. What conclusion with respect to the roots of the given cubic can be drawn from of the resulting equation in the last example.

43. Criterion of the Nature of the Roots of a Cubic. We can from the form of the equation of differences obtained in Art. 42 derive criteria, in terms of the co-efficients, of the nature of the roots of the algebraical cubic. For, when the equation (5) of Art. 42 has a negative root, the cubic (3) must have a pair of imaginary roots in order that the square of their difference should be negative ; and when (5) has no negative root, the cubic (3) has all its roots real, since a pair of imaginary roots of (3) would give rise to a negative root of (5).

In what follows it is assumed that the co-efficients of the equation are real quantities. Four cases may be distinguished :—

(1) *When G^2+4H^3 is negative, the roots of the cubic are all real.*— For, to make this negative H must be negative (and $4H^3>G^2$) ; the signs of the equation (5) are then alternately positive and negative, and, therefore, (Art. 20), (5) has no negative root ; and consequently the given cubic has all its roots real.

(2) *When G^2+4H^3 is positive, the cubic hast wo imaginary roots.*— For the equation (5) must then have a negative root.

(3) *When $G^2+4H^3=0$, the cubic has two equal roots.*—For the equation (5) has then one root equal to zero. In this case $\triangle=0$, it being assumed that a_0 does not vanish. We may say, therefore, that the *vanishing of the discriminant* (Art. 42) *expresses the condition for equal roots.*

(4) *When $G=0$, and $H=0$, the cubic has its three roots equal.*— For the roots of (5) are then all equal to zero. These equations may also be expressed, as can be easily seen, in the form

$$\frac{a_0}{a_1}=\frac{a_1}{a_2}=\frac{a_2}{a_3},$$

which relations among the co-efficients are, therefore, the *conditions that the cubic should be a perfect cube.*

44. Equation of Differences in General.—The general problem of the formation, by the aid of symmetric functions, of the equation whose roots are the differences, or the squares of the differences, of the roots of a given equation, may be treated as follows :—

Let the proposed equation be

$$f(x)\equiv(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)\dots(x-\alpha_n)=0.$$

Substituting $x+\alpha_r$, for x and giving r the values $1, 2, 3, \dots, n$, in succession, we have the equations

$$\left.\begin{aligned}f(x+\alpha_1) &\equiv x(x+\alpha_1-\alpha_2)(x+\alpha_1-\alpha_3)\ldots(x+\alpha_1-\alpha_n),\\ f(x+\alpha_2) &\equiv x(x+\alpha_2-\alpha_1)(x+\alpha_2-\alpha_3)\ldots(x+\alpha_2-\alpha_n),\\ &\cdots\cdots\cdots\\ f(x+\alpha_n) &\equiv (x+\alpha_n-\alpha_1)(x+\alpha_n-\alpha_2)\ldots(x+\alpha_n-\alpha_{n-1}).\end{aligned}\right\}\ \ldots(1)$$

Also, employing the expansion of Art. 6, and observing that $f(\alpha_r)=0$, we find the equation

$$\frac{1}{x}f(x+\alpha_r)=f'(\alpha_r)+\frac{x}{1.2}f''(\alpha_r)+\frac{x^2}{1.2.3}f'''(\alpha_r)+\ldots+x^{n-1}.$$

Denoting the second side of this equation by $\phi(x,\alpha_r)$, and multiplying both sides of the identities (1), we obtain

$$\phi(x,\alpha_1)\phi(x,\alpha_2)\ldots\phi(x,\alpha_n)\equiv\{x^2-(\alpha_1-\alpha_2)^2\}\{x^2-(\alpha_1-\alpha_3)^2\}\ldots$$
$$\ldots\{x^2-(\alpha_{n-1}-\alpha_n)^2\}.$$

To form the equation of differences, therefore, we can multiply together the n factors $\phi(x,\alpha_1)$, $\phi(x,\alpha_2)$, etc., and substitute for the symmetric functions of the roots which occur in the product their values in terms of the co-efficients. Or we may, as already explained in Art. 42, form directly the product of $\frac{1}{2}n(n-1)$ factors on the right-hand side of the above identity, and express the symmetric functions involved in terms of the co-efficients. The roots of the resulting equation of the $n(n-1)^{th}$ degree in x are equal in pairs with opposite signs. Since in this equation x occurs in even powers only, we may substitute y for x^2, and thus obtain the equation of the $\frac{1}{2}n(n-1)^{th}$ degree whose roots are the squared differences.

For equations beyond the third degree the formation of the equation of differences becomes laborious. We shall give the result in the case of the general algebraic equation of the fourth degree in a subsequent chapter.

Examples

1. The roots of the equation

$$x^3-6x^2+11x-6=0$$

are α, β, γ; form the equation whose roots are

$$\beta^2+\gamma^2,\quad \gamma^2+\alpha^2,\quad \alpha^2+\beta^2.$$

[*Ans.* $y^3-28y^2+245y-650=0$.

2. The roots of the cubic

$$x^3+2x^2+3x+1=0$$

are α, β, γ; form the equation whose roots are

$$\frac{1}{\beta^3}+\frac{1}{\gamma^3}-\frac{1}{\alpha^3},\quad \frac{1}{\gamma^3}+\frac{1}{\alpha^3}-\frac{1}{\beta^3},\quad \frac{1}{\alpha^3}+\frac{1}{\beta^3}-\frac{1}{\gamma^3}.$$

[*Ans.* $y^3+12y^2-172y-2072=0$

3. The roots of the cubic

$$x^3+qx+r=0$$

are α, β, γ; form the equation whose roots are

$$\beta^2+\beta\gamma+\gamma^2, \quad \gamma^2+\gamma\alpha+\alpha^2, \quad \alpha^2+\alpha\beta+\beta^2. \qquad [\textit{Ans.} \ (y+q)^3=0.$$

4. The roots of the cubic

$$x^3+px^2+qx+r=0$$

being α, β, γ; form the equation whose roots are

$$\beta^2+\gamma^2-\alpha^2, \ \gamma^2+\alpha^2-\beta^2, \ \alpha^2+\beta^2-\gamma^2.$$

$$[\textit{Ans.} \ y^3-(p^2-2q)y^2-(p^4-4p^2q+8pr)y+p^6-6p^4q+8p^3r$$
$$+8p^2q^2-16pqr+8r^3=0.$$

5. If α, β, γ be the roots of the cubic

$$x^3-3(1+a+a^2)x+1+3a+3a^2+2a^3=0;$$

prove that $(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)$ is a rational function of a. [*Ans.* $\pm 9(1+a+a^2)$

6. Find the relation between G and H of the cubic

$$a_0x^3+3a_1x^2+3a_2x+a_3=0,$$

when its roots are so related that $(\beta-\gamma)^2$, $(\gamma-\alpha)^2$, $(\alpha-\beta)^2$ are in arithmetical progression. [*Ans.* $G^2+2H^3=0$.

7. If α, β, γ, δ be the roots of

$$c^2x^4-2c^2x^3+2x-1=0,$$

find the value of

$$(\beta^2-\gamma^2)^2(\alpha^2-\delta^2)^2+(\gamma^2-\alpha^2)^2+(\beta^2-\delta^2)^2(\alpha^2-\beta^2)^2(\gamma^2-\delta^2)^2. \qquad [\textit{Ans.} \ 0$$

8. Prove that, if

$$\beta\gamma+\gamma\alpha+\alpha\beta+\alpha\delta+\beta\delta+\gamma\delta=0,$$
$$\{(\beta-\gamma)^2(\alpha-\delta)^2+(\gamma-\alpha)^2(\beta-\delta)^2+(\alpha-\beta)^2(\gamma-\delta)^2\}^2$$
$$=18\{(\beta^2-\gamma^2)^2(\alpha^2-\delta^2)^2+(\gamma^2-\alpha^2)^2(\beta^2-\delta^2)^2+(\alpha^2-\beta^2)^2(\gamma^2-\delta^2)\}^2.$$

9. Solve the equation

$$x^5-x^4+8x^2-9x-15=0,$$

which has one root of the form $1+\alpha\sqrt{-1}$.

Diminish the roots by 1; substitute $\alpha\sqrt{-1}$ for x; we find that α must satisfy $\alpha^4-3\alpha^2-4=0$, and $\alpha^4-6\alpha^2+8=0$; hence $\alpha=\pm 2$. Hence the factor x^2-2x+5. The other factors are (x^2+1) and (x^2-3), as is evident.

10. The roots of the cubic

$$a_0x^3+3a_1x^2+3a_2x+a_3=0,$$

are α, β, γ; form the equation whose roots are

$$\beta+\gamma, \quad \gamma+\alpha, \quad \alpha+\beta.$$

This question has been already solved in Art. 41. We give here another solution which, although in this particular instance it is not the simplest, will be found convenient in many examples. Let the roots of the given equation be diminished by h. The transformed equation is (Art. 35).

$$a_0y^3+3A_1y^2+3A_2y+A_3=0,$$

whose roots are $\alpha-h$, $\beta-h$, $\gamma-h$. We express the condition that this equation should have two roots equal with opposite signs This condition is (*see* Ex. 17, Art. 24)

$$9A_1A_2-a_0A_3=0.$$

This equation is a cubic in h whose roots are

$$\tfrac{1}{2}(\beta+\gamma), \ \tfrac{1}{2}(\gamma+\alpha), \ \tfrac{1}{2}(\alpha+\beta);$$

for the above condition is

$$(\beta-h)+(\gamma-h)=0,$$

or

$$2h=\beta+\gamma,$$

where β, γ represent indifferently any two of the roots. From the equation in h the required cubic can be formed by multiplying the roots by 2.

*11. The roots of the biquadratic

$$a_0x^4+4a_1x^3+6a_2x^2+4a_3x+a_4=0$$

are $\alpha, \beta, \gamma, \delta$; form the sextic whose roots are

$$\beta+\gamma,\ \ \gamma+\alpha,\ \ \alpha+\beta,\ \ \alpha+\delta,\ \ \beta+\delta,\ \ \gamma+\delta.$$

Employing the method of Ex. 10, the required equation can be obtained from the condition of Ex. 20, Art. 24.

The condition is in this case

$$6A_1A_2A_3-A_1^2A_4-a_0A_3^2=0.$$

This is a sextic in h whose roots are $\frac{1}{2}(\beta+\gamma)$, etc., from which the required equation can be obtained as in the last example.

12. Form, for the cubic of Ex. 10, the equation whose roots are

$$\frac{\beta\gamma-\alpha^2}{\beta+\gamma-2\alpha},\ \frac{\gamma\alpha-\beta^2}{\gamma+\alpha-2\beta},\ \frac{\alpha\beta-\gamma^2}{\alpha+\beta-2\gamma}.$$

Diminish the roots by h, and express the condition that the resulting cubic should have its roots in geometric progression (*see* Ex. 18, Art. 24). The condition is

$$A_1^3A_3-a_0A_2^3=0.$$

This will be found to reduce to a cubic in h; whose roots are the values above written, since

$$(\alpha-h)^2=(\beta-h)(\gamma-h),\text{ or }h=\frac{\beta\gamma-\alpha^2}{\beta+\gamma-2\alpha}.$$

13. Form for the same cubic the equation whose roots are

$$\frac{2\beta\gamma-\alpha\beta-\alpha\gamma}{\beta+\gamma-2\alpha},\ \frac{2\gamma\alpha-\beta\gamma-\beta\alpha}{\gamma+\alpha-2\beta},\ \frac{2\alpha\beta-\gamma\alpha-\gamma\beta}{\alpha+\beta-2\gamma}.$$

Diminish the roots by h, and express the condition that the transformed cubic should have its roots in harmonic progression (*see* Ex. 19, Art. 24). We have

$$\frac{2}{\alpha-h}=\frac{1}{\beta-h}+\frac{1}{\gamma-h}$$

or

$$h=\frac{2\beta\gamma-\alpha\beta-\alpha\gamma}{\beta+\gamma-2\alpha}.$$

The equation in h is

$$a_0A_3^2-3A_1A_2A_3+2A_2^3=0,$$

which will be found to reduce to a cubic.

14. The roots of the biquadratic

$$a_0x^4+4a_1x^3+6a_2x^2+4a_3x+a_4=0$$

are $\alpha, \beta, \gamma, \delta$; find the cubic whose roots are

$$\frac{\beta\gamma-\alpha\delta}{\beta+\gamma-\alpha-\delta},\ \frac{\gamma\alpha-\beta\delta}{\gamma+\alpha-\beta-\delta},\ \frac{\alpha\beta-\gamma\delta}{\alpha+\beta-\gamma-\delta}.$$

Diminish the roots by h, and employ the condition of Ex. 22, Art. 24. The condition is in this case

$$A_1^2 A_4 - a_0 A_3^2 = 0,$$

which reduces to a cubic in h whose roots are the values above written.

15. Find the equation whose roots are the ratios of the roots of the cubic

$$x^3 + qx + r = 0.$$

The general problem can be solved by elimination. Let $f(x)=0$ be the given equation, and $\rho = \frac{\beta}{\alpha}$ = the ratio of two roots; then since $f(\beta)=0$, we have $f(\rho\alpha)=0$, also $f(\alpha)=0$; and the required equation in ρ is obtained by eliminating α between these two latter equations. For the cubic in the present example the result is

$$r^2(\rho^2+\rho+1)^3 + q^3\rho^2(\rho+1)^2 = 0.$$

16. If α, β, γ be the roots of

$$x^3 + px^2 + qx + r = 0,$$

form the equation whose roots are

$$\beta^2+\gamma^2,\ \ \gamma^2+\alpha^2,\ \ \alpha^2+\beta^2.$$

[*Ans.* $x^3 - 2(p^2-2q)x^2 + (p^4 - 4p^2q + 5q^2 - 2pr)x - (p^2q^2 - 2p^3r + 4pqr - 2q^3 - r^2) = 0.$

17. Form for the same cubic the equation whose roots are

$$\frac{\beta}{\gamma}+\frac{\gamma}{\beta},\ \ \frac{\gamma}{\alpha}+\frac{\alpha}{\gamma},\ \ \frac{\alpha}{\beta}+\frac{\beta}{\alpha}.$$

[*Ans.* $r^2x^3 - (pqr - 3r^2)x^2 + (p^3r - 5pqr + 3r^2 + q^3)x$
$- (p^2q^2 - 2p^3r + 4pqr - 2q^3 - r^2) = 0.$

18. If α, β, γ be the roots of the cubic

$$x^3 + qx + r = 0,$$

form the equation whose roots are

$$l\alpha + m\beta\gamma,\ \ l\beta + m\gamma\alpha,\ \ l\gamma + m\alpha\beta.$$

[*Ans.* $y^3 - mqy^2 + (l^2q + 3lmr)y + l^3r - l^2mq^2 - 2lm^2qr - m^3r^2 = 0.$

19. If α, β, γ be the roots of the cubic

$$a_0x^3 + 3a_1x^2 + 3a_2x + a_3 = 0,$$

form the equation whose roots are

$$(\alpha-\beta)(\alpha-\gamma),\ (\beta-\gamma)(\beta-\alpha),\ (\gamma-\alpha)(\gamma-\beta).$$

$\left[\textit{Ans.}\ \ y^3 + \frac{9H}{a_0^2}y^2 - \frac{27(G^2+4H^3)}{a_0^6} = 0.\right.$

20. Form, for the cubic of Ex. 19, the equation whose roots are

$$(\beta-\gamma)^2(2\alpha-\beta-\gamma)^2,\ \ (\gamma-\alpha)^2(2\beta-\gamma-\alpha)^2,\ \ (\alpha-\beta)^2(2\gamma-\alpha-\beta)^2.$$

The required equation can be obtained by forming the equation of squared differences of the cubic (4) of Art. 42, since

$$(\gamma-\alpha)^2 - (\alpha-\beta)^2 = (\beta-\gamma)(2\alpha-\beta-\gamma).$$

21. Form, for the cubic of Ex. 16, the equation whose roots are

$$\alpha(\beta-\gamma)^2,\ \beta(\gamma-\alpha)^2,\ \gamma(\alpha-\beta)^2.$$

Let the transformed equation be $x^3 + Px^2 + Qx + R = 0.$

[*Ans.* $P = pq - 9r$, $Q = q^3 - 9pqr + 27r^2 + p^3r.$
$R = -r(4q^3 + 27r^2 + 4p^3r - p^2q^2 - 18pqr).$

22. Form, for the same cubic, the equation whose roots are

$$\alpha^2+2\beta\gamma,\ \ \beta^2+2\gamma\alpha,\ \ \gamma^2+2\alpha\beta.$$

[*Ans.* $P = -p^2$, $Q = q(2p^2 - 3q)$, $-R = 4p^3r - 18pqr + 2q^3 + 27r^2$

CHAPTER V

SOLUTION OF RECIPROCAL AND BINOMIAL EQUATIONS

45. Reciprocal Equations. It has been shown in Art. 32, that all reciprocal equations can be reduced to a standard form, in which the degree is even, and the co-efficients counting from the beginning and end equal with the same sign. We now proceed to prove that *a reciprocal equation of the standard form can always be depressed to another of half the dimensions.*

Consider the equation

$$a_0x^{2m}+a_1x^{2m-1}+\dots\dots+a_mx^m+\dots\dots+a_1x+a_0=0.$$

Dividing by x^m, and uniting terms equally distant from the extreme, we have

$$a_0\left(x^m+\frac{1}{x^m}\right)+a_1\left(x^{m-1}+\frac{1}{x^{m-1}}\right)+\dots\dots+a_{m-1}\left(x+\frac{1}{x}\right)+a_m=0.$$

Assume $x+\dfrac{1}{x}=z$, and let $x^p+\dfrac{1}{x^p}$ be denoted for brevity by V_p. We have plainly the relation

$$V_{p+1}=V_pz-V_{p-1}.$$

Giving in succession the values 1, 2, 3 etc., we have

$$\begin{aligned}
V_2&=V_1z-V_0=z^2-2,\\
V_3&=V_2z-V_1=z^3-3z,\\
V_4&=V_3z-V_2=z^4-4z^2+2;\\
V_5&=V_4z-V_3=z^5-5z^3+5z\ ;
\end{aligned}$$

and so on. Substituting these values in the above equation, we get an equation of the m^{th} degree in z ; and from the values of z those of x can be obtained by solving a quadratic.

Examples

1. Find the roots of the equation

$$x^5+x^4+x^3+x^2+x+1=0.$$

Dividing by $x+1$ (*see* Art. 32), we have

$$x^4+x^2+1=0.$$

This equation may be depressed to the form

$$z^2-1=0, \text{ giving } z=\pm 1$$

whence $$x+\frac{1}{x}=1,\ x+\frac{1}{x}=-1$$

and the roots of these equations are

$$\frac{1\pm\sqrt{-3}}{2},\ \frac{-1\pm\sqrt{-3}}{2}.$$

2. Find the roots of the equation

$$x^{10}-3x^8+5x^6-5x^4+3x^2-1=0$$

Dividing by x^2-1, which may be done briefly as follows (see Art. 8),

$$\begin{array}{cccccc} 1 & -3 & 5 & -5 & 3 & -1 \\ & 1 & -2 & 3 & -2 & 1 \\ \hline & -2 & 3 & -2 & 1 & 0 \end{array}$$

we have the reciprocal equation

$$x^8-2x^6+3x^4-2x^2+1=0, \quad \ldots(1)$$

or

$$\left(x^4+\frac{1}{x^4}\right)-2\left(x^2+\frac{1}{x^2}\right)+3=0.$$

Substituting for V_4, z^4-4z^2+2 ; and for V_2, z^2-2, we have the equation

$$z^4-6z^2+9=0, \text{ or } (z^2-3)^2=0,$$

whence

$$z^2=3, \text{ and } z=\pm\sqrt{3},$$

giving

$$x+\frac{1}{x}=\sqrt{3},\ x+\frac{1}{x}=-\sqrt{3};$$

and the roots of these equations are

$$\frac{\sqrt{3}\pm\sqrt{-1}}{2},\ \frac{-\sqrt{3}\pm\sqrt{-1}}{2}.$$

These roots are double roots of the equation (1).

3. Solve the equation

$$x^5-1=0.$$

Dividing by $x-1$ we have

$$x^4+x^3+x^2+x+1=0;$$

from which we obtain

$$z^2+z-1=0.$$

Solving this equation, we have the quadratics

$$x^2+\tfrac{1}{2}(1+\sqrt{5})x+1=0$$
$$x^2+\tfrac{1}{2}(1-\sqrt{5})x+1=0,$$

from which we obtain

$$x=\tfrac{1}{4}\{-1+\theta\sqrt{5}\pm(10+2\theta\sqrt{5})^{\frac{1}{2}}\ \sqrt{-1}\},$$

where $\theta^2=1$.

This expression gives the four values of x.

4. Find the quadratic factors of

$$x^6+1=0.$$

Transforming this, we have

$$z^3-3z=0.$$

whence

$$z=0, \text{ and } z=\pm\sqrt{3}.$$

The quadratic factors of the given equation are, therefore,

$$x^2+1=0,\ x^2\pm x\sqrt{3}+1=0.$$

5. Solve the equations

(1) $(1+x)^4=a(1+x^4)$, (2) $(1+x)^5=a(1+x^5)$.

6. Reduce to an equation of the fourth degree in z

$$\frac{(1+x)^5}{1+x^5}+\frac{(1-x)^5}{1-x^5}=2a.$$

[*Ans.* $(1-a)z^4+(7+3a)z^2-(4+a)=0$.

46. Binomial Equations. General Properties. In this and the following articles will be proved the leading general properties of binomial equations.

PROP. I.—*If α be an imaginary root of $x^n-1=0$, then α^m also will be a root, m being any integer.*

Since α is a root,

$$\alpha^n=1, \text{ and, therefore, } (\alpha^n)^m=1, \text{ or } (\alpha^m)^n=1\ ;$$

that is, $\qquad \alpha^m$ is a root of $x^n-1=0$.

The same is true of the equation $x^n+1=0$, except that in this case m must be an *odd* integer.

47. PROP. II.—*If m and n be prime to each other, the equations $x^m-1=0$, $x^n-1=0$ have no common root except unity.*

To prove this we make use of the following property of numbers :—*If m and n be integers prime to each other, integers a and b can be found such that $mb-na=\pm 1$.* For, in fact, when $\frac{m}{n}$ is turned into a continued fraction, $\frac{a}{b}$ is the approximation preceding the final restoration of $\frac{m}{n}$.

Now, if possible, let α be any common root of the given equation ; then

$$\alpha^m=1, \text{ and } \alpha^n=1\ ;$$

therefore $\qquad \alpha^{mb}=1, \text{ and } \alpha^{na}=1\ ;$

whence $\qquad \alpha^{(mb-na)}=1, \text{ or } \alpha^{\pm 1}=1, \text{ or } \alpha=1\ ;$

that is, 1 is the only root common to the given equations.

48. PROP. III.—*If k be the greatest common measure of two integers m and n, the roots common to the equations $x^m-1=0$, and $x^n-1=0$, are roots of the equation $x^k-1=0$.*

To prove this, let

$$m=km',\ n=kn'.$$

Now, since m' and n' are prime to each other, integers b and a may be found such that $m'b-n'a=\pm 1$; hence

$$mb-na=\pm k.$$

If, therefore, α be a common root of $x^m-1=0$, and $x^n-1=0$,

$$\alpha^{(mb-na)}=1, \text{ or } \alpha^k=1\ ;$$

which proves that α is a root of the equation $x^k-1=0$.

49. Prop. IV.—*When n is a prime number, and α any imaginary root of $x^n-1=0$, all the roots are included in the series*

$$1,\ \alpha,\ \alpha^2, \ldots,\ \alpha^{n-1}$$

For by Prop. I, these quantities are all roots of the equation. And they are all different ; for, if possible, let any two of them be equal,

$$\alpha^p=\alpha^q,$$

whence

$$\alpha^{(p-q)}=1\ ;$$

but, by Prop. II, this equation is impossible, since n is necessarily prime to $(p-q)$, which is a number less than n.

50. Prop. V.—*When n is a composite number formed of the factors p, q, r, etc., the roots of the equations $x^p-1=0$, $x^q-1=0$, $x^r-1=0$, etc., all satisfy the equation $x^n-1=0$.*

For, consider a root α of the equation $x^p-1=0$; then $\alpha^p=1$; from which we derive

$$(\alpha^p)^{qr}\ldots=1\ ;\ \text{ or } \alpha^n-1=0\ ;$$

which proves the proposition.

51. Prop. VI.—*When n is a composite number formed of the prime factors p, q, r, etc., the roots of the equation $x^n-1=0$ are the n terms of the product*

$$(1+\alpha+\alpha^2+\ldots+\alpha^{p-1})(1+\beta+\ldots+\beta^{q-1})(1+\gamma+\ldots+\gamma^{r-1})\ldots\ldots,$$

where α is a root of $x^p-1=0$, β of $x^q-1=0$, γ of $x^r-1=0$, etc.

We prove this for the case of three factors p, q, r. A similar proof applies in general. Any term, *e.g.*, $\alpha^a\beta^b\gamma^c$, of the product is evidently a root of the equation $x^n-1=0$, since $\alpha^{an}=1$, $\beta^{bn}=1$, $\gamma^{cn}=1$, and, therefore, $(\alpha^a\beta^b\gamma^c)^n=1$. And no two terms of the product can be equal ; for, if possible let $\alpha^a\beta^b\gamma^c$ be equal to another term $\alpha^{a'}\beta^{b'}\gamma^{c'}$; then $\alpha^{a-a'}=\beta^{b-b'}=\gamma^{c-c'}$. The first member of this equation is a root of $x^p-1=0$, and the second member is a root of $x^{qr}-1=0$. Now these two equations cannot have a common root since p and qr are prime to each other (Prop. II) ; hence $\alpha^a\beta^b\gamma^c$ cannot be equal to $\alpha^{a'}\beta^{b'}\gamma^{c'}$.

52. Prop. VII.—*The roots of the equation $x^n-1=0$, where $n=p^aq^br^c$, and p, q, r, are the prime factors of n, are the n products of the form $\alpha\beta\gamma$, where α is a root of $x^{p^a}=1$, β a root of $x^{q^b}=1$, and γ of $x^{r^c}=1$.*

This is an extension of Prop. VI to the case where the prime factors occur more than once in n. The proof is exactly similar.

Any such product $\alpha\beta\gamma$ must be a root, since $\alpha^n=1$, $\beta^n=1$, $\gamma^n=1$, n being a multiple of p^a, q^b, r^c; and a proof similar to that of Art. 51 shows that no two such products can be equal, since p^a, q^b, r^c are prime to one another. We have, for convenience, stated this proposition for three factors only of n. A similar proof can be applied to the general case.

From this and the preceding propositions we are now able to derive the following general conclusion .—

The determination of the n^{th} roots of unity is reduced to the case where n is a prime number, or a power of a prime number.

53. The Special Roots of the Equation $x^n-1=0$. Every equation $x^n-1=0$ has certain roots which do not belong to any equation of similar form and lower degree. Such roots we call *special roots** of that equation, or *special n^{th} roots of unity.* If n be a prime number, all the imaginary roots are roots of this kind. If $n=p^a$, where p is a prime number, any n^{th} root of a lower degree than n must belong to the equation $x^{p^{a-1}}-1=0$, since every divisor of p^a is a divisor of p^{a-1} (except n itself); hence there are $p^a\left(1-\frac{1}{p}\right)$ roots which belong to no lower degree. If, again, $n=p^aq^b$, where p and q are prime to each other, there are $p^a\left(1-\frac{1}{p}\right)$, and $q^b\left(1-\frac{1}{q}\right)$ special roots of $x^{p^a}-1=0$, and $x^{q^b}-1=0$, respectively. Now, if α and β be any two special roots of these equations, $\alpha\beta$ is a special root of $x^n-1=0$; for if not, suppose $(\alpha\beta)^m=1$, where m is less than n; we have then $\alpha^m=\beta^{-m}$; but α^m is a root of $x^{p^a}-1=0$, and β^{-m} is a root of $x^{q^b}-1=0$, and these equations cannot have a common root other than 1, as their degrees are prime to each other; consequently m cannot be less than n, and $\alpha\beta$ is a special root of $x^n-1=0$. Also, as there are

$$p^a\left(1-\frac{1}{p}\right)q^b\left(1-\frac{1}{q}\right), \text{ or } n\left(1-\frac{1}{p}\right)\left(1-\frac{1}{q}\right),$$

such products, there are the same number of special n^{th} roots. This proof may be extended without difficulty to any form of n.

All the roots of $x^n-1=0$ are given by the series $1, \alpha, \alpha^2, \ldots, \alpha^{n-1}$; where α is any special n^{th} root. For it is plain that α, α^2 etc., are all roots. And no two are equal; for, if $\alpha^p=\alpha^q$, $\alpha^{(p-q)}=1$; and, therefore, α is not a special n^{th} root, since $p-q$ is less than n.

*The term "special root" is here used in preference to the usual term "primitive root," since the latter has a different signification in the theory of numbers.

When one special n^{th} root α is given, we may obtain all the other special n^{th} roots of unity.

Since α is a special root, all the roots $1, \alpha, \alpha^2, \ldots, \alpha^{n-1}$ are different n^{th} roots, as we have just proved; and if we select a root α of this series, where p is prime to n, the roots

$$\alpha^p, \alpha^{2p}, \ldots, \alpha^{(n-1)p}, \alpha^{np}(=1)$$

are all different, since the exponents of α when divided by n give different remainders in every case; that is, the series of numbers $0, 1, 2, 3, \ldots\ldots, n-1$ in some order; whence this series of roots is the same as the former, except that the terms occur in a different order. To each number p prime to n and less than it (1 included), corresponds a special n^{th} root of unity; for α^{mp} cannot be equal to 1 when m is less than n, for if it were we should have two roots in the series equal to 1, and the series could not give all the roots in that case; therefore, α^p is not a root of any binomial equation of a degree inferior to n; that is, α^p is a special n^{th} root of unity. What is here proved agrees with the result above established, since the number of integers less than n and prime to it is, by a known property of numbers, $n\left(1-\dfrac{1}{p}\right)\left(1-\dfrac{1}{q}\right)$ when $n=p^a q^b$, which is also, as above proved, the number of special roots of $x^n-1=0$.

Examples

1. To determine the special roots of $x^6-1=0$.

Here $6=2\times 3$. Consequently the roots of the equations $x^3-1=0$, and $x^2-1=0$ are roots of $x^6-1=0$. Now, dividing x^6-1 by x^3-1 we have x^3+1; and dividing x^3+1 by $\dfrac{x^2-1}{x-1}$, or $x+1$, we have $x^2-x+1=0$, which determines the special roots of $x^6-1=0$.

Solving the quadratic, the roots are

$$\alpha=\frac{1+\sqrt{-3}}{2}, \quad \alpha_1=\frac{1-\sqrt{-3}}{2};$$

also since

$$\alpha\alpha_1=1=\alpha^6,$$
$$\alpha_1=\alpha^5,$$

which may be easily verified.

The special roots are, therefore,

$$\alpha, \alpha^5; \text{ or } \alpha_1{}^5, \alpha_1; \text{ or } \alpha, \frac{1}{\alpha}.$$

2. To discuss the special roots of $x^{12}-1=0$.

Since 2 and 3 are the prime factors of 12, and $\dfrac{12}{2}=6$, $\dfrac{12}{3}=4$, the roots of $x^6-1=0$, and $x^4-1=0$, are roots of $x^{12}-1=0$; now, dividing $x^{12}-1$ by x^4-1,

and x^6-1, and equating the quotients to zero, we have the two equations $x^8+x^4+1=0$, and $x^6+1=0$, both of which may be satisfied by the special roots of $x^{12}-1=0$; therefore, taking the greatest common measure of x^8+x^4+1, and x^6+1, and equating it to zero, the special roots are the roots of the equation $x^4-x^2+1=0$.

The same result would plainly have been arrived at by dividing $x^{12}-1$ by the least common multiple of x^4-1 and x^6-1. Now, solving the reciprocal equation $x^4-x^2+1=0$, we have $x+\dfrac{1}{x}=\pm\sqrt{3}$; whence, if α and α_1 be two special roots

$$\left(\alpha,\ \frac{1}{\alpha}\right)=\frac{\sqrt{3}\pm\sqrt{-1}}{2},\quad \left(\alpha_1,\ \frac{1}{\alpha_1}\right)=\frac{-\sqrt{3}\pm\sqrt{-1}}{2}$$

are the four special roots of $x^{12}-1=0$.

We proceed now to express the four special roots in terms of any one of them α.

Since $\quad \alpha+\dfrac{1}{\alpha}+\alpha_1+\dfrac{1}{\alpha_1}=0$, or $(\alpha+\alpha_1)\left(1+\dfrac{1}{\alpha\alpha_1}\right)=0$,

we take $\alpha\alpha_1=-1$ (as consistent with the values we have assigned to α and α_1); and since α and α_1 are roots of $x^6+1=0$, $\alpha^6=-1$, and $\alpha^5=-\dfrac{1}{\alpha}=\alpha_1$. The roots $\alpha,\ \alpha_1,\ \dfrac{1}{\alpha_1},\ \dfrac{1}{\alpha}$ may, therefore, be expressed by the series $\alpha, \alpha^5, \alpha^7, \alpha^{11}$, since $\alpha^{12}=1$.

Further, replacing α by $\alpha^5, \alpha^7, \alpha^{11}$, we have, including the series just determined, the four following series, by omitting multiples of 12 in the exponents of α:—

$$\begin{array}{cccc} \alpha, & \alpha^5 & \alpha^7 & \alpha^{11} \\ \alpha^5, & \alpha, & \alpha^{11}, & \alpha^7, \\ \alpha^7, & \alpha^{11}, & \alpha, & \alpha^5, \\ \alpha^{11}, & \alpha^7, & \alpha^5, & \alpha, \end{array}$$

where the same roots are reproduced in every row and column, their order only being changed. We have, therefore, proved that this property is not peculiar to any one root of the four special roots; and it will be noticed, in accordance with what is above proved, in general, that 1, 5, 7 and 11 are all the numbers prime to 12, and less than it. We may obtain all the roots of $x^{12}-1=0$ by the powers of any one of the four special roots $\alpha, \alpha^5, \alpha^7, \alpha^{11}$, as follows:—

$$\begin{array}{cccccccccccc} \alpha, & \alpha^2, & \alpha^3, & \alpha^4, & \alpha^5, & \alpha^6, & \alpha^7, & \alpha^8, & \alpha^9, & \alpha^{10}, & \alpha^{11}, & 1, \\ \alpha^5, & \alpha^{10}, & \alpha^3, & \alpha^8, & \alpha, & \alpha^6, & \alpha^{11}, & \alpha^4, & \alpha^9, & \alpha^2, & \alpha^7, & 1, \\ \alpha^7, & \alpha^2, & \alpha^9, & \alpha^4, & \alpha^{11}, & \alpha^6, & \alpha, & \alpha^8, & \alpha^3, & \alpha^{10}, & \alpha^5, & 1, \\ \alpha^{11}, & \alpha^{10}, & \alpha^9, & \alpha^8, & \alpha^7, & \alpha^6, & \alpha^5, & \alpha^4, & \alpha^3, & \alpha^2, & \alpha. & 1 \end{array}$$

3. Prove that the special roots of $x^{15}-1=0$ are roots of the equation

$$x^8-x^7+x^5-x^4+x^3-x+1=0.$$

4. Show that the eight roots of the equation in the preceding example may be obtained by multiplying the two roots of $x^2+x+1=0$ by the four roots of

$$x^4+x^3+x^2+x+1=0.$$

5. Form the equation of the 12th degree whose roots are the special roots of $x^{21}-1=0$, and reduce it to one half of the dimensions.

[*Ans.* $x^6-x^5-6x^4+6x^3+8x^2-8x+1=0$.

54. Solution of Binomial Equations by Circular Functions. We take the most general binomial equation

$$x^n = a + b\sqrt{-1},$$

where a and b are real quantities.

Let $$a = R\cos\alpha,\ b = R\sin\alpha;$$

then $$x^n = R(\cos\alpha + \sqrt{-1}\sin\alpha);$$

now, if $$r(\cos\theta + \sqrt{-1}\sin\theta)$$

be a root of this equation, we have, by De Moivre's Theorem,

$$r^n(\cos n\theta + \sqrt{-1}\sin n\theta) = R(\cos\alpha + \sqrt{-1}\sin\alpha);$$

and, therefore,

$$r^n\cos n\theta = R\cos\alpha,$$

$$r^n\sin n\theta = R\sin\alpha.$$

Squaring these two equalities, and adding,

$$r^{2n} = R^2, \text{ giving } r^n = R;$$

where we take R and r both positive, since in expressions of the kind here considered the factor containing the angle may always be taken to involve the sign.

We have then

$$\cos n\theta = \cos\alpha,\ \sin n\theta = \sin\alpha;$$

and, consequently,

$$n\theta = \alpha + 2k\pi,$$

k being any integer; whence the assumed n^{th} root is of the general type

$$\sqrt[n]{R}\left(\cos\frac{\alpha+2k\pi}{n} + \sqrt{-1}\sin\frac{\alpha+2k\pi}{n}\right).$$

Giving to k in this expression any n consecutive values in the series of numbers between $-\infty$ and $+\infty$, we get all the n^{th} roots, and no more than n, since the n values recur in periods.

We may write the expression for the n^{th} root under the form

$$\left\{\sqrt[n]{R}\left(\cos\frac{\alpha}{n} + \sqrt{-1}\sin\frac{\alpha}{n}\right)\right\}\left(\cos\frac{2k\pi}{n} + \sqrt{-1}\sin\frac{2k\pi}{n}\right).$$

If we now suppose $R=1$, and $\alpha=0$, the equation $x^n = a + b\sqrt{-1}$ becomes $x^n = 1 + 0\sqrt{-1}$; the general type, therefore, of an n^{th} root of $1 + 0\sqrt{-1}$, or unity, is

$$\cos\frac{2k\pi}{n} + \sqrt{-1}\sin\frac{2k\pi}{n}.$$

If we give k any definite value, for instance zero,

$$\sqrt[n]{R}\left(\cos\frac{\alpha}{n}+\sqrt{-1}\sin\frac{\alpha}{n}\right)$$

is one n^{th} root of $a+b\sqrt{-1}$.

The preceding formula shows, therefore, that *all the n^{th} roots of any imaginary quantity may be obtained by multiplying any one of them by the n^{th} roots of unity.*

Taking in conjunction the binomial equations

$$x^n=a+b\sqrt{-1}, \text{ and } x^n=a-b\sqrt{-1},$$

we see that the factors of the trinomial

$$x^{2n}-2R\cos\alpha.x^n+R^2$$

are

$$\sqrt[n]{R}\left\{\cos\frac{\alpha+2k\pi}{n}\pm\sqrt{-1}\sin\frac{\alpha+2k\pi}{n}\right\},$$

where k has the values $0, 1, 2, 3, \ldots, n-1$.

Examples

1. Solve the equation $x^7-1=0$.

Dividing by $x-1$, this is reduced to the standard form of reciprocal equation.

Assuming $z=x+\frac{1}{x}$, we obtain the cubic

$$z^3+z^2-2z-1=0,$$

from whose solution that of the required equation is obtained.

2. Resolve $(x+1)^7-x^7-1$ into factors. [*Ans.* $7x(x+1)(x^2+x+1)^2$

3. Find the quintic on whose solution that of the binomial equation $x^{11}-1=0$ depends. [*Ans.* $z^5+z^4-4z^3-3z^2+3z+1=0$.

4. When a binomial equation is reduced to the standard form of reciprocal equation (by division by $x-1$, $x+1$, or x^2-1), show that the reduced equation has all its roots imaginary. (*Cf.* Examples 15, 16, p. 26).

5. When this reduced reciprocal equation is transformed by the substitution $z=x+\frac{1}{x}$; show that the equation in z has all its roots real, and situated between -2 and 2.

For the roots of the equation in x are of the form $\cos\alpha+\sqrt{-1}\sin\alpha$ (*see* Art. 54); hence $x+\frac{1}{x}$ is of the form $2\cos\alpha$, and the value of this is real and between -2 and 2.

6. Show that the following equation is reciprocal, and solve it :—

$$4(x^2-x+1)^3-27x^2(x-1)^2=0.$$

[*Ans.* Roots : $2, 2, \frac{1}{2}, \frac{1}{2}, -1, -1$.

7. Exhibit all the roots of the equation $x^9-1=0$.

The solution of this is reduced to the solution of the three cubics

$$x^3-1=0, \quad x^3-\omega=0, \quad x^3-\omega^2=0;$$

where, ω, ω^2 are the imaginary cube roots of unity. The nine roots may be represented as follows:—

$$1,\ \omega^{\frac{1}{3}},\ \omega^{\frac{2}{3}},\ \omega,\ \omega^{\frac{4}{3}},\ \omega^{\frac{5}{3}},\ \omega^2,$$

Excluding 1, ω, ω^2, the other six roots are special roots of the given equation; and are the roots of the sextic

$$x^6+x^3+1=0.$$

8. Reducing the equation of the 8^{th} degree in Ex. 3, Art. 53, by the substitution $z=x+\dfrac{1}{x}$, we obtain

$$z^4-z^3-4z^2+4z+1=0$$

prove that the roots of this equation are

$$2\cos\frac{2\pi}{15},\ 2\cos\frac{4\pi}{15},\ 2\cos\frac{8\pi}{15},\ 2\cos\frac{15\pi}{15}.$$

9. Reduce the equation

$$4x^4-85x^3+357x^2-340x+64=0$$

to a reciprocal equation, and solve it.

Assume $$z=\frac{x}{2}+\frac{2}{x}.$$ [*Ans.* Roots : $\frac{1}{4}$, 1, 4, 16

10 Solve the equation

$$x^4+mpx^3+m^2qx^2+m^3px+m^4=0.$$

Dividing the roots by m, this reduces to a reciprocal equation.

11. If α be an imaginary root of the equation $x^n-1=0$, where n is a prime number; prove the relation

$$(1-\alpha)(1-\alpha^2)(1-\alpha^3)......(1-\alpha^{n-1})=n.$$

12. Show that a cubic equation can be reduced immediately to the reciprocal form when the relation of Ex. 18, Art. 24, exists amongst its co-efficients.

13. Show that a biquadratic can be reduced immediately to the reciproca form when the relation of Ex. 22, Art. 24, exists among its co-efficients.

14. Form the cubic whose roots are

$$\alpha+\alpha^6,\ \alpha^3+\alpha^4,\ \alpha^2+\alpha^5,$$

where α is an imaginary root of $x^7-1=0$. [*Ans.* $x^3+x^2-2x-1=0$.

When the roots of this cubic are known, the solution of the equation $x^7-1=0$ may be completed by means of quadratics. For, suppose the three root to be x_1, x_2, x_3; then α and α^6 are the roots of $x^2-x_1x+1=0$; α^3 and α^4 of $x^2-x_2x+1=0$, and α^2 and α^5 of $x^2-x_3x+1=0$. It is easy to see that the roots of the cubic are all real, and they may be readily found approximately by the methods of Chap. X.

15. Form the cubic whose roots are

$$\alpha+\alpha^8+\alpha^{12}+\alpha^5,\ \alpha^2+\alpha^3+\alpha^{11}+\alpha^{10},\ \alpha^4+\alpha^6+\alpha^9+\alpha^7,$$

where α is an imaginary root of $x^{13}-1=0$. [*Ans.* $x^3+x^2-4x+1=0$.

As in the preceding example, when the roots of the cubic (which are all real) are known, the solution of the binomial equation $x^{13}-1=0$ may be completed by solving quadratics. Let x_1, x_2, x_3 be the roots of the cubic. It is easily seen that $\alpha+\alpha^{12}$ and $\alpha^8+\alpha^5$ are the roots of $x^2-x_1x+x_3=0$; $\alpha^2+\alpha^{11}$ and $\alpha^3+\alpha^{10}$ of

$x^2-x_2x+x_1=0$, and $\alpha^4+\alpha^9$ and $\alpha^6-\alpha^7$ of $x^2-x_3x+x_2=0$. When these quadratics are solved, each pair of roots α, α^{12} : α^8, α^5, etc., may be found by the solution of another quadratic, as in the preceding example.

16. Reduce to quadratics the solution of $x^{17}-1=0$.

Calling α one of the imaginary roots, we form the quadratic whose roots are

$$\alpha_1 \equiv \alpha+\alpha^9+\alpha^{13}+\alpha^{15}+\alpha^{16}+\alpha^8+\alpha^4+\alpha^2,$$
$$\alpha_2 \equiv \alpha^3+\alpha^{10}+\alpha^5+\alpha^{11}+\alpha^{14}+\alpha^7+\alpha^{12}+\alpha^6.$$

We easily find $\alpha_1\alpha_2=4(\alpha_1+\alpha_2)=-4$; hence α_1 and α_2 are the roots of $x^2+x-4=0$, and may be found by solving this quadratic. Assuming, again,

$$\left.\begin{array}{l}\beta_1 \equiv \alpha+\alpha^{13}+\alpha^{16}+\alpha^4, \\ \beta_2 \equiv \alpha^9+\alpha^{15}+\alpha^8+\alpha^2,\end{array}\right\} \left.\begin{array}{l}\gamma_1 \equiv \alpha^3+\alpha^5+\alpha^{14}+\alpha^{12}, \\ \gamma_2 \equiv \alpha^{10}+\alpha^{11}+\alpha^7+\alpha^6,\end{array}\right\}$$

it is seen that β_1, β_2 are the roots of $x^2-\alpha_1x-1=0$, and γ_1, γ_2 of $x^2-\alpha_2x-1=0$. Separating again each of these into two parts, and forming the quadratic whose roots are, for example, $\alpha+\alpha^{16}$ and $\alpha^{13}+\alpha^4$, the sums of the roots in pairs are obtained; and finally the roots themselves, by the solution of quadratics, as in the preceding examples.

This and the preceding two are examples of Gauss's method of solving algebraically the binomial equation $x^n-1=0$ when n is a prime number. The solution of such an equation can be made to depend on the solution of equations of degree not higher than the greatest prime number which is a factor in $n-1$. When $n=13$, *e.g.*, the solution depends on that of a cubic, $n-1$ being $=3 \cdot 2^2$ in that case; and when $n=17$, the solution is reducible in quadratics, $n-1$ being then $=2^4$. For the application of Gauss's method it is necessary to arrange the $n-1$ imaginary roots in a suitable order in each case according to the powers of any one of them. A "primitive root" of a prime number n possesses the property that when raised to successive powers from 0 to $n-2$ inclusive, and divided in each case by n, the $n-1$ remainders are all different. (*See* Serret's *Cours d' Algebre Superieure,* Vol. II. Sec 3). There are several such primitive roots of any prime number; *e.g.*, 2, 6, 7, and 11 of 13, and 3, 5, 6, 7, 10, 11, 12 14 of 17. Gauss arranges the imaginary roots so that the successive indices of any one of them, α, are the successive powers from 0 to $n-2$ of any primitive root of n. Taking, for example, the lowest primitive root of 13, and dividing the successive powers of 2 by 13, we get the following series of remainders—

1 2 4 8 3 6 12 11 9 5 10 7;

and these, therefore, are the successive powers of α in order when the indices which exceed 13 are reduced by the equation $\alpha^{13}=1$. If the lowest primitive root of 17 be treated in the same way, we get the following series of remainders :—

1 3 9 10 13 5 15 11 16 14 8 7 4 12 2 6.

On comparing these series with the assumptions above made, it will be observed that in the former case, *viz*, $n=13$, the twelve roots were divided into three sums of four each, and in the latter case into two sums of eight each. The method of partition in any case depends on the nature of the factors of $n-1$; and it is not difficult to show in general that the product of any two such groups is equal to the sum of two or more, as the student will have observed in the particular applications given above.

The lowest primitive root in any particular case is the only one necessary to be known for the application of Gauss's method; and this can usually

be found without difficulty by trial. It may be observed that one or other of the three simplest prime numbers 2, 3, 5, is a primitive root in the case of every prime number less than 100, with exception of 41 and 71, whose lowest primitive roots are 6 and 7 respectively. Methods of finding all the primitive roots are given in the section of Sorret's work above referred to.

17. Find by trial the lowest primitive root of 19, and hence show how to solve the equation $x^{19}-1=0$.

It is readily found that 2 is a positive root, and the remainders after division by 19 are given in the process of trial. Since $18=3^2.2$, the solution will be effected by cubics and quadratics. The first cubic is found by forming the equation whose roots are

$$\alpha+\alpha^8+\alpha^7+\alpha^{18}+\alpha^{11}+\alpha^{12},$$
$$\alpha^2+\alpha^{16}+\alpha^{14}+\alpha^{17}+\alpha^3+\alpha^5,$$
$$\alpha^4+\alpha^{13}+\alpha^9+\alpha^{15}+\alpha^6+\alpha^{10}.$$

18. Show that of binomial equations whose degree is a prime number the lowest after $x^{17}-1=0$ whose solution depends on quadratics is $x^{257}-1=0$.

The next prime number after 257 which satisfies the condition that $n-1$ is a power of 2 is 65537. We have, therefore, the series 3, 5, 17, 257, 65537, etc., and Gauss remarks (*Disquisitiones Arithmeticae*, Art. 365) that the division of a circle into n equal parts, or the description of a regular polygon of n sides can be effected by geometrical constructions when n has any of these values.

19. If $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ be the roots of the equation

$$x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots+p_{n-1}x+p_n=0,$$

form the equation whose roots are

$$\alpha_1+\frac{1}{\alpha_1},\ \alpha_2+\frac{1}{\alpha_2}, \ldots,\ \alpha_n+\frac{1}{\alpha_n}.$$

We have here the identity

$$x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots+p_{n-1}x+p_n\equiv(x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_n);$$

and changing x into $\frac{1}{x}$ (*see* Art. 32),

$$p_nx^n+p_{n-1}x^{n-1}+\ldots+p_2x^2+p_1x+1\equiv p_n\left(x-\frac{1}{\alpha_1}\right)\left(x-\frac{1}{\alpha_2}\right)\ldots\left(x-\frac{1}{\alpha_n}\right).$$

Multiplying together these identities, and dividing by x^n, the factors on the right-hand side take the form $x+\frac{1}{x}-\left(\alpha+\frac{1}{\alpha}\right)$; and assuming $x+\frac{1}{x}=z$, the left-hand side can be expressed as a polynomial of the n^{th} degree in z by means of the relations of Art. 45.

20. Find the value of symmetric function $\Sigma\alpha^2\beta^2(\gamma-\delta)^2$ of the roots of the equation

$$a_0x^4+4a_1x^3+6a_2x^2+4a_3x+a_4=0.$$

This can be derived from the result of Ex. 19, p. 41, by changing the roots into their reciprocals, forming $\Sigma\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)^2$ of the transformed equation, and multiplying by $\alpha^2\beta^2\gamma^2\delta^2$, which is equal to $\frac{a_4^2}{a_0^2}$.

[*Ans.* $a_0^2\Sigma\alpha^2\beta^2(\gamma-\delta)^2=48(a_3^2-a_2a_4)$.

From the values of the symmetric functions given in Chapter III several others can be obtained by the process here indicated.

21. Find the value of the symmetric function $\Sigma(\alpha_1-\alpha_2)^2\,\alpha_3^{\,2}\alpha_4^{\,2}\ldots\alpha_n^{\,2}$ of the roots of the equation

$$a_0x^n+na_1x^{n-1}+\frac{n(n-1)}{1.2}a_2x^{n-2}+\ldots+na_{n-1}x+a_n=0.$$

We easily obtain $a_0^{\,2}\Sigma(\alpha_1-\alpha_2)^2=n^2(n-1)(a_1^{\,2}-a_0a_2)$; and changing the roots into their reciprocals we have

$$a_0^{\,2}\Sigma(\alpha_1-\alpha_2)^2\,\alpha_3^{\,2}\alpha_4^{\,2}\ldots\alpha_n^{\,2}=n^2(n-1)(a_{n-1}^{\,2}-a_{n-2}a_n).$$

22. Show that the five roots of the equation

$$x^5-5px^3+5p^2x+2q=0$$

are

$$\sqrt[5]{a}+\sqrt[5]{b},\ \theta\sqrt[5]{a}+\theta^4\sqrt[5]{b},\ \theta^2\sqrt[5]{a}+\theta^3\sqrt[5]{b},$$
$$\theta^4\sqrt[5]{a}+\theta\sqrt[5]{b},\ \theta^3\sqrt[5]{a}+\theta^2\sqrt[5]{b},$$

where $\sqrt[5]{ab}=p$, $a+b=-2q$, and θ is an imaginary fifth root of unity.

N.B.—A quintic reducible to this form can consequently be immediately solved.

23. Write down trigonometrical expressions for the roots in the preceding example; and, p being supposed essentially positive, prove—

(1) when $p^5<q^2$, the roots are one real and four imaginary;

(2) when $p^5>q^2$, the roots are all real;

(3) when $p^5=q^2$, there is a square quadratic factor.

24. Find the following product, where θ is an imaginary fifth root of unity:—

$$(\alpha+\beta+\gamma)(\alpha+\theta\beta+\theta^4\gamma)(\alpha+\theta^2\beta+\theta^3\gamma)(\alpha+\theta^3\beta+\theta^2\gamma)(\alpha+\theta^4\beta+\theta\gamma).$$

[*Ans.* $\alpha^5+\beta^5+\gamma^5-5\alpha\beta\gamma(\alpha^2-\beta\gamma)$.

25. Form the biquadratic equation whose roots are

$$\alpha+2\alpha^4,\ \alpha^2+2\alpha^3,\ \alpha^3+2\alpha^2,\ \alpha^4+2\alpha,$$

where α is an imaginary root of $x^5-1=0$. [*Ans.* $x^4+3x^3-x^2-3x+11=0$.

CHAPTER VI

ALGEBRAIC SOLUTION OF THE CUBIC AND BIQUADRATIC

55. On the Algebraic Solution of Equations. Before proceeding to the solution of cubic and biquadratic equations we make some introductory remarks, with a view of putting clearly before the student the general principles on which the algebraic solution of these equations depends. With this object we give in the present Article three methods of solution of the quadratic, and state as we proceed how these methods may be extended to cubic and biquadratic equations, leaving the subsequent Articles the complete development of the principles involved.

(1) *First method of solution—by assuming for a root a general form involving radicals.*

Since the expression $p+\sqrt{q}$ has two, and only two, values when the square root involved is taken with the double sign, this is a natural form to take for the root of a quadratic. Assuming, therefore, $x=p+\sqrt{q}$, and rationalizing, we have

$$x^2-2px+p^2-q=0.$$

Now, if this be identical with a given quadratic equation

$$x^2+Px+Q=0,$$

we have $$2p=-P,\ p^2-q=Q,$$

giving $$x=p+\sqrt{q}=\frac{-P\pm\sqrt{P^2-4Q}}{2},$$

which is the solution of the quadratic.

In the case of the cubic equation we shall find that

$$\sqrt[3]{p}+\frac{A}{\sqrt[3]{p}},\ \text{and}\ \sqrt[3]{p}\sqrt[3]{q}(\sqrt[3]{p}+\sqrt[3]{q})$$

are both proper forms to represent a root, these expressions having each three, and only three, values when the cube roots involved are taken in all generality.

In the case of the biquadratic equation we shall find that

$$\sqrt{p}+\sqrt{q}+\frac{A}{\sqrt{p}\sqrt{q}},\quad \sqrt{q}\sqrt{r}+\sqrt{r}\sqrt{p}+\sqrt{p}\sqrt{q}$$

are forms which may represent a root, these expressions each giving four, and only four, values of x when the square roots receive their double signs.

(2) *Second method of solution—by resolving into factors.*

Let it be required to resolve the quadratic x^2+Px+Q into its imple factors. For this purpose we put it under the form

$$x^2+Px+Q+\theta-\theta,$$

and determine θ so that

$$x^2+Px+Q+\theta$$

may be a perfect square, *i.e.*, we make

$$\theta+Q=\frac{P^2}{4}, \text{ or } \theta=\frac{P^2-4Q}{4};$$

whence, putting for θ its value, we have

$$x^2+Px+Q\equiv\left(x+\frac{P}{2}\right)^2-\left(0x+\frac{\sqrt{P^2-4Q}}{2}\right)^2$$

Thus we have reduced the quadratic to the form u^2-v^2; and its simple factors are $u+v$, and $u-v$.

Subsequently we shall reduce the cubic to the form

$$(lx+m)^3-(l'x+m')^3, \text{ or } u^3-v^3,$$

and obtain its solution from the simple equations

$$u-v=0,\ u-\omega v=0,\ u-\omega^2 v=0.$$

It will be shown also that the biquadratic may be reduced to either of the forms

$$(lx^2+mx+n)^2-(l'x^2+m'x+n')^2,$$
$$(x^2+px+q)(x^2+p'x+q'),$$

by solving a cubic equation; and, consequently, the solution of the biquadratic completed by solving two quadratics, *viz.*, in the first case, $lx^2+mx+n=\pm(l'x^2+m'x+n')$; and in the second case,

$$x^2+px+q=0, \text{ and } x^2+p'x+q'=0.$$

(3) *Third method of solution—by symmetric functions of the roots.*

Consider the quadratic equation $x^2+Px+Q=0$, of which the roots are α, β. We have the relations

$$\alpha+\beta=-P,$$
$$\alpha\beta=Q.$$

If we attempt to determine α and β by these equations, we fall back on the original equation (*see* Art. 24); but if we could obtain a second equation between the roots and co-efficients, of the form

$l\alpha+m\beta=f(P, Q)$, we could easily find α and β by means of this equation and the equation $\alpha+\beta=-P$.

Now in this case of the quadratic there is no difficulty in finding the required equation ; for, obviously,

$$(\alpha-\beta)^2=P^2-4Q \text{ ; and, therefore, } \alpha-\beta=\sqrt{P^2-4Q}.$$

In the case of the cubic equation $x^3+Px^2+Qx+R=0$, we require *two* simple equations of the form

$$l\alpha+m\beta+n\gamma=f(P, Q, R),$$

in addition to the equation $\alpha+\beta+\gamma=-P$, to determine the roots α, β, γ. It will subsequently be proved that the functions

$$(\alpha+\omega\beta+\omega^2\gamma)^3, (\alpha+\omega^2\beta+\omega\gamma)^3$$

may be expressed in terms of the co-efficients by showing the *quadratic* equation ; and when their values are known the roots of the cubic may be easily found.

In this case of the biquadratic equation

$$x^4+Px^3+Qx^2+Rx+S=0$$

we require *three* simple equations of the form

$$l\alpha+m\beta+n\gamma+r\delta=f(P, Q, R, S),$$

in addition to the equation,

$$\alpha+\beta+\gamma+\delta=-P,$$

to determine the roots $\alpha, \beta, \gamma, \delta$. It will be proved in Art. 66, that the three functions

$$(\beta+\gamma-\alpha-\delta)^2, (\gamma+\alpha-\beta-\delta)^2, (\alpha+\beta-\gamma-\delta)^2$$

may be expressed in terms of the co-efficients by solving a *cubic* equation ; and when their values are known the roots of the biquadratic equation may be immediately obtained.

56. The Algebraic Solution of the Cubic Equation. Let the general cubic equation

$$ax^3+3bx^2+3cx+d=0$$

be put under the form

$$z^3+3Hz+G=0.$$

where

$$z=ax+b, H=ac-b^2, G=a^2d-3abc+2c^3 \qquad \text{(Art. 36)}$$

To solve this equation, assume*

$$z=\sqrt[3]{p}+\sqrt[3]{q} \text{ ;}$$

*This solution is usually called *Cardan's solution of the cubic. See* note *A* at the end of the volume.

hence, cubing,

$$z^3=p+q+3\sqrt[3]{p}\sqrt[3]{q}(\sqrt[3]{p}+\sqrt[3]{q})$$

therefore,

$$z^3-3\sqrt[3]{p}\sqrt[3]{q}\,.\,z-(p+q)=0.$$

Now, comparing co-efficients, we have

$$\sqrt[3]{p}\sqrt[3]{q}=-H,\ p+q=-G\ ;$$

from which equations we obtain

$$p=\tfrac{1}{2}(-G+\sqrt{G^2+4H^3}),\ q=\tfrac{1}{2}(-G-\sqrt{G^2+4H^3})\ ;$$

and, substituting for $\sqrt[3]{q}$ its value $\dfrac{-H}{\sqrt[3]{p}}$, we have

$$z=\sqrt[3]{p}+\frac{-H}{\sqrt[3]{p}}$$

as the algebraic solution of the equation

$$z^3+3Hz+G=0.$$

It should be noted that if p be replaced by q this value of z is unchanged, as the terms are then simply interchanged ; also, since $\sqrt[3]{p}$ has the three values $\sqrt[3]{p}$, $\omega\sqrt[3]{p}$, $\omega^2\sqrt[3]{p}$, obtained by multiplying any one of its values by the three cube roots of unity, we obtain three, and only three, values for z, namely,

$$\sqrt[3]{p}+\frac{-H}{\sqrt[3]{p}},\ \omega\sqrt[3]{p}+\omega^2\frac{-H}{\sqrt[3]{p}},\ \omega^2\sqrt[3]{p}+\omega\frac{-H}{\sqrt[3]{p}}\ ;$$

the order of these values only changing according to the cube root of p selected.

Now, if z be replaced by its value $ax+b$, we have, finally,

$$ax+b=\sqrt[3]{p}+\frac{-H}{\sqrt[3]{p}}$$

(where p has the value previously determined in terms of co-efficients) as the *complete algebraic solution of the cubic equation*

$$ax^3+3bx^2+3cx+d=0,$$

the square root and cube root involved being taken in their entire generality.

57. Application to Numerical Equations. The solution of the cubic which has been obtained, unlike the solution of the quadratic, is of little practical value when the co-efficients of the equation are given numbers ; although as an algebraic solution it is complete.

For, when the roots of the cubic are real, $G^2+4H^3=-K^2$, an essentially negative number (*see* Art. 43); and, substituting for p and q their values

$$\tfrac{1}{2}(-G\pm\sqrt{-1})$$

in the formula $\sqrt[3]{p}+\sqrt[3]{q}$, we have the following expression for a root of the cubic :—

$$\left(\frac{-G+K\sqrt{-1}}{2}\right)^{\frac{1}{3}}+\left(\frac{-G-K\sqrt{-1}}{2}\right)^{\frac{1}{3}}.$$

Now there is no general arithmetical process for extracting the cube root of such complex numbers, and consequently this formula is useless for purposes of arithmetical calculation.

But when the cubic has a pair of imaginary roots, a numerical value may be obtained from the formula

$$\left(\frac{-G+\sqrt{G^2-4H^3}}{2}\right)^{\frac{1}{3}}+\left(\frac{-G-\sqrt{G^2+4H^3}}{2}\right)^{\frac{1}{3}},$$

since G^2+4H^3 is positive in this case. As a practical method, however, of obtaining the real root of a numerical cubic, this process is of little value.

In the first case, namely, where the roots are all real, we can make use of Trigonometry to obtain the numerical values of the roots in the following manner :—

Assuming $2R\cos\phi=-G$, and $2R\sin\phi=K$, we have

$$p=Re^{\phi\sqrt{-1}},\ q=Re^{-\phi\sqrt{-1}};$$

also
$$\tan\phi=-\frac{K}{G},\ \text{and}\ R=\tfrac{1}{2}(G^2+K^2)^{\frac{1}{2}}=(-H)^{\frac{3}{2}};$$

and finally, since $\omega=\cos\frac{2\pi}{3}\pm\sqrt{-1}\sin\frac{2\pi}{3}=e^{\pm\frac{2\pi}{3}\sqrt{-1}}$,

the three roots of the cubic equation

$$z^3+3Hz+G=0,$$

viz.,
$$\sqrt[3]{p}+\sqrt[3]{q},\ \omega\sqrt[3]{p}+\omega^2\sqrt[3]{q},\ \omega^2\sqrt[3]{p}+\omega\sqrt[3]{q},$$

become
$$2(-H)^{\frac{1}{2}}\cos\frac{\phi}{3},\quad -2(-H)^{\frac{1}{2}}\cos\frac{\pi\pm\phi}{3};$$

from which formulas we obtain the numerical values of the roots of the cubic by aid of a table of sines and cosines. This process is not convenient in practice ; and in general, for purposes of arithmetical calculation of real roots, the methods of solution of numerical equations to be hereafter explained (Chapter X) should be employed.

58. Expression of the Cubic as the Difference of two Cubes. Let the given cubic

$$ax^3+3bx^2+3cx+d\equiv\phi(x)$$

be put under the form

$$z^3+3Hz+G,$$

where $z \equiv ax+b$.

Now, assuming

$$z^3+3Hz+G \equiv \frac{1}{\mu-\nu}\{\mu(z+\nu)^3-\nu(z+\mu)^3\}, \qquad \ldots(1)$$

where μ and ν are quantities to be determined, the second side of this identity becomes, when reduced,

$$z^3-3\mu\nu z-\mu\nu(\mu+\nu).$$

Comparing co-efficients,

$$\mu\nu=-H,\ \mu\nu(\mu+\nu)=-G\ ;$$

therefore

$$\mu+\nu=\frac{G}{H},\ \mu-\nu=\frac{a\sqrt{\Delta}}{H}$$

where $a^2\Delta \equiv G^2+4H^3$, as in Art. 42,

also $$(z+\mu)\,(z+\nu) \equiv z^2+\frac{G}{H}z-H. \qquad \ldots(2)$$

Whence, putting for z its value, $ax+b$, we have from (1)

$a^3\phi x \equiv$

$$\left(\frac{G+a\Delta^{\frac{1}{2}}}{2\Delta^{\frac{1}{2}}}\right)\left(ax+b+\frac{G-a\Delta^{\frac{1}{2}}}{2H}\right)^3-\left(\frac{G-a\Delta^{\frac{1}{2}}}{2\Delta^{\frac{1}{2}}}\right)\left(ax+b+\frac{G+a\Delta^{\frac{1}{2}}}{2H}\right)^3$$

which is the required expression for $\phi(x)$ as the difference of two cubes.

By the aid of the identity just proved the cubic can be resolved into its simple factors, and the solution of the equation completed. We proceed to obtain expressions for the roots of the equation $\phi(x)=0$ in terms of μ and ν. Solving as a binomial cubic the equation

$$(\mu-\nu)a^2\phi(x) \equiv \mu(z+\nu)^3-\nu(z+\mu)^3=0,$$

we find the three following values for $z \equiv ax+b$:—

$$\sqrt[3]{\mu}\ \sqrt[3]{\nu}(\sqrt[3]{\mu}+\sqrt[3]{\nu}),$$
$$\sqrt[3]{\mu}\ \sqrt[3]{\nu}(\omega\sqrt[3]{\mu}+\omega^2\sqrt[3]{\nu}),$$
$$\sqrt[3]{\mu}\ \sqrt[3]{\nu}(\omega^2\sqrt[3]{\mu}+\omega\sqrt[3]{\nu}).$$

If now $\sqrt[3]{\mu}$ and $\sqrt[3]{\nu}$ be replaced by any pair of cube roots selected one from each of the two series

$$\sqrt[3]{\mu},\ \omega\sqrt[3]{\mu},\ \omega^2\sqrt[3]{\mu},$$
$$\sqrt[3]{\nu},\ \omega\sqrt[3]{\nu},\ \omega^2\sqrt[3]{\nu},$$

it will be seen that we shall get the same three values of z, the *order* only of these values changing according to the cube roots selected. It follows that the expression

$$\sqrt[3]{\mu}\,\sqrt[3]{\nu}(\sqrt[3]{\mu}+\sqrt[3]{\nu})$$

has three, and only three, values when the cube roots therein are taken in all generality. This form, therefore, is, in addition to that obtained in the last Article, a form proper to represent a root of a cubic equation [*see* (1), Art. 55].

The function (2) given above, when transformed and reduced, becomes, as may be easily seen,

$$\frac{a^2}{H}\{(ac-b^2)x^2+(ad-bc)x+(bd-c^2)\}.$$

This quadratic, therefore, contains as factors the two binomials $ax+b+\mu$, $ax+b+\nu$, which occur in the above expression of $\phi(x)$ as the difference of two cubes.

59. Solution of the Cubic by Symmetric Functions of the Roots. Since the three values of the expression

$$\tfrac{1}{3}\{\alpha+\beta+\gamma+\theta(\alpha+\omega\beta+\omega^2\gamma)+\theta^2(\alpha+\omega^2\beta+\omega\gamma)\},$$

when θ takes the values 1, ω, ω^2, are α, β, γ, it is plain that if the functions

$$\theta(\alpha+\omega\beta+\omega^2\gamma), \quad \theta^2(\alpha+\omega^2\beta+\omega\gamma)$$

were expressed in terms of the co-efficients of the cubic, we could by substituting their values in the formula given above, arrive at an algebraical solution of the cubic equation. Now this cannot be done directly by solving a quadratic equation ; for, although the product of the two functions above written is a rational symmetric function of α, β, γ, their sum is not so. It will be found, however, that the sum of the cubes of the two functions in question is a symmetric function of the roots, and can, therefore, be expressed by the co-efficients, as we proceed to show. For convenience we adopt the notation

$$L\equiv\alpha+\omega\beta+\omega^2\gamma, \quad M\equiv\alpha+\omega^2\beta+\omega\gamma.$$

We have then

$$(\theta L)^3=A+B\omega+C\omega^2, \quad (\theta^2 M)^3=A+B\omega^2+C\omega,$$

where

$$A=\alpha^3+\beta^3+\gamma^3+6\alpha\beta\gamma, \quad B=3(\alpha^2\beta+\beta^2\gamma+\gamma^2\alpha), \quad C=3(\alpha\beta^2+\beta\gamma^2+\gamma\alpha^2),$$

from which we obtain

$$L^3+M^3=2\Sigma\alpha^3-3\Sigma\alpha^2\beta+12\alpha\beta\gamma=-27\frac{G}{a^3}$$

(*Cf.* Ex. 5, p. 35 : Ex. 15, p. 40).

Again,

$$(\theta L)(\theta^2 M) = LM = \alpha^2+\beta^2+\gamma^2-\beta\gamma-\gamma\alpha-\alpha\beta = -9\frac{H}{a^2},$$

whence $$(\alpha+\omega\beta+\omega^2\gamma)^3,\ (\alpha+\omega^2\beta+\omega\gamma)^3$$
are the roots of the quadratic equation

$$t^2+3^3\frac{G}{a^3}t-3^6\frac{H^3}{a^6}=0.$$

Denoting the roots of this equation, *viz.*,

$$\frac{3^3}{2a^3}\left(-G\pm\sqrt{G^2+4H^3}\right)$$

by t_1 and t_2, the original formula expressed in terms of the co-efficients of the cubic gives for the three roots

$$\alpha = -\frac{b}{a}+\frac{1}{3}\left(\sqrt[3]{t_1}+\sqrt[3]{t_2}\right),$$

$$\beta = -\frac{b}{a}+\frac{1}{3}\left(\omega\sqrt[3]{t_1}+\omega^2\sqrt[3]{t_2}\right),$$

$$\gamma = -\frac{b}{a}+\frac{1}{3}\left(\omega^2\sqrt[3]{t_1}+\omega\sqrt[3]{t_2}\right).$$

It will be seen that the values of α, β, γ here arrived at are of the same form as those already obtained in Art. 56.

It is important to observe that the functions

$$(\alpha+\omega\beta+\omega^2\gamma)^3,\ (\alpha+\omega^2\beta+\omega\gamma)^3$$

are remarkable as being the simplest functions of *three* quantities which have but *two* values when these quantities are interchanged in every way. It is owing to this property that the solution of a cubic equation can be reduced to that of a quadratic. Several functions of α, β, γ of this nature exist, and it will be proved in a subsequent chapter that any two such functions are connected by a rational linear relation in terms of the co-efficients.

Having now completed the discussion of the different modes of algebraical solution of the cubic, we give some examples involving the principles contained in the preceding Articles.

Examples

1. Resolve into simple factors the expression

$$(\beta-\gamma)^2(x-\alpha)^2+(\gamma-\alpha)^2(x-\beta)^2+(\alpha-\beta)^2(x-\gamma)^2.$$

Let $U=(\beta-\gamma)(x-\alpha)$, $V=(\gamma-\alpha)(x-\beta)$, $W=(\alpha-\beta)(x-\gamma)$.

[*Ans.* $\frac{2}{3}(U+\omega V+\omega^2 W)(U+\omega^2 V+\omega W)$.

2. Prove that the several equations of the system

$$(\beta-\gamma)^3(x-\alpha)^3=(\gamma-\alpha)^3(x-\beta)^3=(\alpha-\beta)^3(x-\gamma)^3$$

have two factors common.

Making use of the notation in the last example, we have

$$U^3=V^3=W^3;$$

whence

$$U^3-V^3=(U-V)(U^2+UV+V^2)\equiv\tfrac{1}{2}(U-V)(U^2+V^2+W^2)$$

since

$$U+V+W\equiv 0;$$

therefore,

$$(\beta-\gamma)^2(x-\alpha)^2+(\gamma-\alpha)^2(x-\beta)^2+(\alpha-\beta)^2(x-\gamma)^2$$

is the common quadratic factor required.

3. Resolve into factors the expressions

(1) $(\beta-\gamma)^3(x-\alpha)^3+(\gamma-\alpha)^3(x-\beta)^3+(\alpha-\beta)^3(x-\gamma)^3$,

(2) $(\beta-\gamma)^5(x-\alpha)^5+(\gamma-\alpha)^5(x-\beta)^5+(\alpha-\beta)^5(x-\gamma)^5$,

(3) $(\beta-\gamma)^7(x-\alpha)^7+(\gamma-\alpha)^7(x-\beta)^7+(\alpha-\beta)^7(x-\gamma)^7$.

These factors can be written down at once from the results established in Ex. 40, p. 47. Using the notation of Ex. 1, and replacing α_1, β_1, γ_1, in the example referred to, by U, V, W, we obtain the following:

[*Ans.* (1) $3UVW$; (2) $\frac{5}{2}(U^2+V^2+W^2)UVW$; (3) $\frac{7}{4}(U^2+V^2+W^2)^2\,UVW$.

4. Express

$$(x-\alpha)(x-\beta)(x-\gamma)$$

as the difference of two cubes.

Assume

$$(x-\alpha)(x-\beta)(x-\gamma)=U_1^3-V_1^3$$

whence

$$U_1-V_1=\lambda(x-\alpha),$$
$$\omega U_1-\omega^2V_1=\mu(x-\beta),$$
$$\omega^2U_1-\omega V_1=\nu(x-\gamma).$$

Adding, we have

$$\lambda+\mu+\nu=0,\ \lambda\alpha+\mu\beta+\nu\gamma=0;$$

and, therefore,

$$\lambda=\rho(\beta-\gamma),\ \mu=\rho(\gamma-\alpha),\ \nu=\rho(\alpha-\beta);$$

but $\lambda\mu\nu=1$; whence

$$\frac{1}{\rho^3}=(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta).$$

Substituting these values of λ, μ, ν; and using the notation of Ex. 1.

$$U_1-V_1=\rho U,\ \omega U_1-\omega^2V_1=\rho V,\ \omega^2U_1-\omega V_1=\rho W;$$

whence

$$3U_1=\rho(U+\omega^2V+\omega W),$$
$$-3V_1=\rho(U+\omega V+\omega^2W);$$

and U_1 and V_1 are completely determined.

5. Prove that L and M are functions of the differences of the roots.

We have $\qquad L=\alpha+\omega\beta+\omega^2\gamma=\alpha-h+\omega(\beta-h)+\omega^2(\gamma-h)$

for all values of h, since $1+\omega+\omega^2=0$; and giving to h the values α, β, γ in succession, we obtain three forms for L in terms of the differences $\beta-\gamma$, $\gamma-\alpha$, $\alpha-\beta$. Similarly for M.

6. To express the product of the squares of the differences of the roots in terms of the co-efficients.

We have

$$L+M=2\alpha-\beta-\gamma,\ L+\omega^2M=(2\beta-\gamma-\alpha)\omega,\ L+\omega M=(2\gamma-\alpha-\beta)\omega^2\ ;$$

and, again,

$$L-M=(\beta-\gamma)(\omega-\omega^2),\ \omega^2L-\omega M=(\gamma-\alpha)(\omega-\omega^2),\ \omega L-\omega^2M=(\alpha-\beta)(\omega-\omega^2),$$

from which we obtain, as in Art. 26,

$$L^3+M^3=(2\alpha-\beta-\gamma)(2\beta-\gamma-\alpha)(2\gamma-\alpha-\beta),$$
$$L^3-M^3=-3\sqrt{-3}(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)\ ;$$

and since

$$(L^3-M^3)^2\equiv(L^3+M^3)^2-4L^3M^3,$$

we have, substituting the values of L^3+M^3 and LM obtained in Art. 59,

$$a^6(\beta-\gamma)^2(\gamma-\alpha)^2(\alpha-\beta)^2=-27(G^2+4H^3)$$

(*Cf.* Art. 42.)

7. Prove the following identities :—

$$L^3+M^3\equiv\tfrac{1}{3}[(2\alpha-\beta-\gamma)^3+(2\beta-\gamma-\alpha)^3+(2\gamma-\alpha-\beta)^3],$$
$$L^3-M^3\equiv\sqrt{-3}[(\beta-\gamma)^3+(\gamma-\alpha)^3+(\alpha-\beta)^3].$$

These are easily obtained by cubing and adding the values of

$$L+M,\ \text{etc.}\ ;\ L-M,\ \text{etc.},$$

in the preceding example.

8. To obtain expressions for L^2, M^2, etc., in terms of the differences of β, γ.

The following forms for L^2 and M^2 are obtained by subtracting $(\alpha^2+\beta^2+\gamma^2)(1+\omega+\omega^2)=0$ from $(\alpha+\omega\beta+\omega^2\gamma)^2$, and $(\alpha+\omega^2\beta+\omega\gamma)^2$:—

$$-L^2=(\beta-\gamma)^2+\omega^2(\gamma-\alpha)^2+\omega(\alpha-\beta)^2,$$
$$-M^2=(\beta-\gamma)^2+\omega(\gamma-\alpha)^2+\omega^2(\alpha-\beta)^2.$$

In a similar manner, we find from these expressions

$$-L^4=(\beta-\gamma)^2(2\alpha-\beta-\gamma)^2+\omega(\gamma-\alpha)^2(2\beta-\gamma-\alpha)^2+\omega^2(\alpha-\beta)^2(2\gamma-\alpha-\beta)^2,$$
$$-M^4=(\beta-\gamma)^2(2\alpha-\beta-\gamma)^2+\omega^2(\gamma-\alpha)^2(2\beta-\gamma-\alpha)^2+\omega(\alpha-\beta)^2(2\gamma-\alpha-\beta)^2.$$

Also, without difficulty, we have the following forms for LM and L^2M^2 :—

$$2LM=(\beta-\gamma)^2+(\gamma-\alpha)^2+(\alpha-\beta)^2,$$
$$L^2M^2=(\alpha-\beta)^2(\alpha-\gamma)^2+(\beta-\gamma)^2(\beta-\alpha)^2+(\gamma-\alpha)^2(\gamma-\beta)^2.$$

9. There are six functions of the type of L or M, *viz.*,

$$\alpha+\omega\beta+\omega^2\gamma,\ \omega\alpha+\omega^2\beta+\gamma,\ \omega^2\alpha+\beta+\omega\gamma,$$
$$\alpha+\omega^2\beta+\omega\gamma,\ \omega\alpha+\beta+\omega^2\gamma,\ \omega^2\alpha+\omega\beta+\gamma,$$

to form the equation whose roots are these six quantities.

These functions may be expressed as follows :—

$$\begin{array}{ccc} L, & \omega L, & \omega^2L, \\ M, & \omega M, & \omega^2M\ ; \end{array}$$

hence they are the roots of the equation

$$(\phi-L)(\phi-\omega L)(\phi-\omega^2L)(\phi-M)(\phi-\omega M)(\phi-\omega^2M)=0,$$

or

$$\phi^6-(L^3+M^3)\phi^3+L^3M^3=0.$$

Substituting for L and M from the equations

$$LM=-\frac{9H}{a^2},\quad L^3+M^3=-27\frac{G}{a^3},$$

we have this equation expressed in terms of the co-efficients as follows :—

$$\phi^6+3^3\frac{G}{a^3}\phi^3-3^6\frac{H^3}{a^6}=0.$$

10. To form, in terms of L and M, the equation whose roots are the squares of the differences of the roots of the general cubic equation.

Let

$$\phi = (\alpha - \beta)^2 ;$$

hence, by former results,

$$\sqrt{-3\phi} = \omega L - \omega^2 M.$$

Retaionalizing this, we obtain

$$\phi(\phi - LM)^2 + \frac{(L^3 - M^3)^2}{27} = 0,$$

which is the required equation.

In a similar manner, by the aid of the results of Ex. 8, the equation of squared differences of this equation, or the equation whose roots are

$$(\beta-\gamma)^2(2\alpha-\beta-\gamma)^2,\ (\gamma-\alpha)^2(2\beta-\gamma-\alpha)^2,\ (\alpha-\beta)^2(2\gamma-\alpha-\beta)^2,$$

is obtained by substituting $-L^2$ and $-M^2$ for M and L, respectively, in the last equation ; and this process may be repeated any number of times. Finally, all these equations may be easily expressed in terms of the co-efficients of the cubic by means of the relations

$$LM = -9\frac{H}{a^2}, \text{ and } L^3 + M^3 = -27\frac{G}{a^3}$$

For instance, the first equation is

$$\phi\left(\phi + 9\frac{H}{a^2}\right)^2 + 27\frac{G^2 + 4H^3}{a^6} = 0.$$

(*Cf.* Art. 42.)

11. If α, β, γ and α', β', γ' be the roots of the cubic equations

$$ax^3 + 3bx^2 + 3cx + d = 0,$$

$$a'x^3 + 3b'x^2 + 3c'x + d' = 0 ;$$

to form the equation which has for roots the six values of the function

$$\phi \equiv \alpha\alpha' + \beta\beta' + \gamma\gamma'.$$

The easiest mode of procedure is first to form the corresponding equation for the cubics deprived of their second terms, *viz.*,

$$z^3 + 3Hz + G = 0,\ z^3 + 3H'z + G' = 0,$$

and thence deduce the equation in the general case ; for in the case of the cubics so transformed the corresponding function

$$\phi_0 \equiv (a\alpha + b)(a'\alpha' + b') + (a\beta + b)(a'\beta' + b') + (a\gamma + b)(a'\gamma' + b')$$
$$\equiv aa'\,\phi - 3bb'.$$

Substituting for the roots of the transformed equations their values expressed by radicals, we have

$$\phi_0 = (\sqrt[3]{p} + \sqrt[3]{q})(\sqrt[3]{p'} + \sqrt[3]{q'}) + (\omega\sqrt[3]{p} + \omega^2\sqrt[3]{q})(\omega\sqrt[3]{p'} + \omega^2\sqrt[3]{q'})$$
$$+ (\omega^2\sqrt[3]{p} + \omega\sqrt[3]{q})(\omega^2\sqrt[3]{p'} + \omega\sqrt[3]{q'}),$$

which reduces to

$$\phi_0 - 3(\sqrt[3]{pq'} + \sqrt[3]{p'q}).$$

Cubing this, we find

$$\varphi_0^3 - 27\sqrt[3]{pqp'q'}\,\varphi_0 - 27(pq' + p'q) = 0.$$

Now, substituting for p and q, p' and q', their values given by the equations,

$$x^2 + Gx - H^3 = 0,\ x^2 + G'x - H'^3 = 0,$$

we have the six values of φ_0 given by the two cubic equations

$$\varphi_0^3 - 27HH'\varphi_0 - \frac{27}{2}(GG' \pm aa'\sqrt{\Delta\Delta'}) = 0,$$

where

$$a^2\Delta = G^2 + 4H^3, \text{ and } a'^2\Delta' = G'^2 + 4H'^3.$$

Finally, substituting for φ_0 its value $aa'\varphi - 3bb'$, and multiplying these cubics together, we have the required equation. It may be noticed that if one of the cubics be $x^3 - 1 = 0$, $\varphi = \alpha + \omega\beta + \omega^2\gamma$, etc., which case has been already considered in Ex. 9. (Mr. M. Roberts, *Dublin Exam. Papers, 1855*)

12. Form the equation whose roots are the several values of ρ, where

$$\rho = \frac{\alpha - \beta}{\alpha - \gamma},$$

and α, β, γ are the roots of the equation $ax^3 + 3bx^2 + 3cx + d = 0$.

Since ρ involves only the differences, and the ratios of α, β, γ, the result will be the same if α, β, γ be replaced by the roots z_1, z_2, z_3 of the equation $z^3 + 3Hz + G = 0$. We have, therefore,

$$(2\rho - 1)z_1 = -(\rho + 1)z_2,$$

$$G = z_1 z_2(z_1 + z_2) = \frac{(2\rho - 1)(\rho - 2)}{(\rho + 1)^2} z_1^3,$$

and similarly
$$H = -\frac{(\rho^2 - \rho + 1)}{(\rho + 1)^2} z_1^2;$$

whence, eliminating z_1, the required equation is

$$H^3[(\rho + 1)(\rho - 2)(2\rho - 1)]^2 + G^2(\rho^2 - \rho + 1)^3 = 0.$$

13. Find the relation between the co-efficients of the cubics

$$ax^3 + 3bx^2 + 3cx + d = 0,$$

$$a'x'^3 + 3b'x'^2 + 3c'x' + d' = 0,$$

when the roots are connected by the equation

$$\alpha(\beta' - \gamma') + \beta(\gamma' - \alpha') + \gamma(\alpha' - \beta') = 0.$$

Multiplying by $\omega - \omega^2$, this equation becomes

$$LM' = L'M.$$

Cubing, and introducing the co-efficients, we find

$$G^2H'^3 = G'^2H^3,$$

the required relation.

14. Determine the condition in terms of the roots and co-efficients that the cubics of Ex. 13 should become identical by the linear transformation

$$x' = px + q.$$

In this case

$$\alpha' = p\alpha + q,\ \beta' = p\beta + q,\ \gamma' = p\gamma + q.$$

Eliminating p and q, we have

$$\beta\gamma'-\beta'\gamma+\gamma\alpha'-\gamma'\alpha+\alpha\beta'-\alpha'\beta=0,$$

which is the function of the roots considered in the last example. This relation, moreover, is unchanged if for α, β, γ ; α', β', γ', we substitute

$$\begin{array}{lll} l\alpha+m, & l\beta+m, & l\gamma+m, \\ l'\alpha'+m', & l'\beta'+m' & l'\gamma'+m' ; \end{array}$$

whence we may consider the cubics in the last example under the simple forms

$$z^3+3Hz+G=0,\ z'^3+3H'z'+G'=0,$$

obtained by the linear transformations $z=ax+b$, $z'=a'x'+b'$; for if the condition holds for the roots of the former equations, it must hold for the roots of the latter. Now putting $z'+kz$, these equations become identical if

$$H'\equiv k^2H,\ G'\equiv k^3G ;$$

whence, eliminating k,

$$G^2H'^3=G'^2H^3$$

is the required condition, the same as that obtained in Ex. 13. It may be observed that the reducing quadratics of the cubics necessarily become identical by the same transformation, *viz.*,

$$\frac{H'}{G'}(a'x'+b')=\frac{H}{G}(ax+b).$$

60. Homographic Relation between two Roots of a Cubic. Before proceeding to the discussion of the biquadratic we prove the following important proposition relative to the cubic :—

The roots of the cubic are connected in pairs by a homographic relation in terms of the coefficients.

Referring to Exs. 13, 14, Art. 27, we have the relations

$$a_0^2\{\ (\beta-\gamma)^2+(\gamma-\alpha)^2+(\alpha-\beta)^2\}=18(a_1^2-a_0a_2),$$

$$a_0^2\{\ \alpha(\beta-\gamma)^2+\beta(\gamma-\alpha)^2+\gamma(\alpha-\beta)^2\}=9(a_0a_3-a_1a_2),$$

$$a_0^2\{\alpha^2(\beta-\gamma)^2+\beta^2(\gamma-\alpha)^2+\gamma^2(\alpha-\beta)^2\}=18(a_2^2-a_1a_3\}.$$

Using the notation

$$a_0a_2-a_1^2\equiv H,\ a_0a_3-a_1a_2\equiv 2H_1,\ a_1a_3-a_2^2\equiv H_2 ;$$

multiplying the above equations by $\alpha\beta$, $-(\alpha+\beta)$, 1, respectively, and adding , since,

$$\alpha^2-\alpha(\alpha+\beta)+\alpha\beta\equiv 0,\ \beta^2-\beta(\alpha+\beta)+\alpha\beta\equiv 0$$

we have

$$a_0^2(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)^2=18\{H\alpha\beta+H_1(\alpha+\beta)+H_2\} ;$$

but

$$a_0^4(\beta-\gamma)^2(\gamma-\alpha)^2(\alpha-\beta)^2=-27\,\Delta\equiv 108(HH_2-H_1^2)$$

(*see* Art. 42) ; whence

$$\pm\sqrt{-\frac{\Delta}{3}}\left(\frac{\alpha-\beta}{2}\right)=H\alpha\beta+H_1(\alpha+\beta)+H_2,$$

and, therefore,

$$H\alpha\beta+\left(H_1+\frac{1}{2}\sqrt{-\frac{\Delta}{3}}\right)\alpha+\left(H_1-\frac{1}{2}\sqrt{-\frac{\Delta}{3}}\right)\beta+H_2=0,$$

which is the required homographic relation. It is to be observed that the coefficients in this equation involve one irrational quantity, the second sign of which will give the relation between a different pair of the roots.

61. First Solution by Radicals of the Biquadratic. Euler's Assumption. Let the biquadratic equation

$$ax^4+4bx^3+6cx^2+4dx+e=0$$

be put under the form (Art. 37)

$$z^4+6Hz^2+4Gz+a^2I-3H^2=0,$$

where $z\equiv ax+b$,

$$H\equiv ac-b^2,\quad I\equiv ae-4bd+3c^2,\quad G\equiv a^2d-3abc+2b^3.$$

To solve this equation (a biquadratic wanting the second term) Euler assumes as the general expression for a root

$$z=\sqrt{p}+\sqrt{q}+\sqrt{r}.$$

Squaring,

$$z^2-p-q-r=2(\sqrt{q}\sqrt{r}+\sqrt{r}\sqrt{p}+\sqrt{p}\sqrt{q}).$$

Squaring again, and reducing, we obtain the equation

$$z^4-2(p+q+r)z^2-8z\sqrt{p}\sqrt{q}\sqrt{r}+(p+q+r)^2-4(qr+rp+pq)=0.$$

Comparing this equation with the former, we have

$$p+q+r=-3H,\ qr+rp+pq=3H^2-\frac{a^2I}{4},\ \sqrt{p}\sqrt{q}\sqrt{r}=-\frac{G}{2};$$

and consequently p, q, r are the roots of the equation

$$t^3+3Ht^2+\left(3H^2-\frac{a^2I}{4}\right)t-\frac{G^2}{4}=0; \qquad \dots(1)$$

or, since

$$-G^2\equiv 4H^3-a^2HI+a^3J, \qquad \text{(Art. 37)},$$

where

$$J\equiv ace+2bcd-ad^2-eb^2-c^3,$$

this equation may be written in the form

$$4(t+H)^3-a^2I(t+H)+a^3J=0;$$

and finally, putting $t+H\equiv a^2\theta$, we obtain the equation

$$4a^3\theta^3-Ia\theta+J=0. \qquad \dots(2)$$

This is called *the reducing cubic* of the biquadratic equation; and will in what follows be referred to by that name. When it is necessary to make a distinction between equations (1) and (2), we shall refer to the former as Euler's cubic.

Also, since $t \equiv b^2 - ac + a^2\theta$; if θ_1, θ_2, θ_3 be the roots of the reducing cubic, we have

$$p \equiv b^2 - ac + a^2\theta_1,\ q \equiv b^2 - ac + a^2\theta_2,\ r \equiv b^2 - ac + a^2\theta_3\ ;$$

and, therefore,

$$z = \sqrt{b^2 - ac + a^2\theta_1} + \sqrt{b^2 - ac + a^2\theta_2} + \sqrt{b^2 - ac + a^2\theta_3}.$$

If this formula be taken to represent a root of the biquadratic in z, it must be observed that the radicals involved have not complete generality; for if they had, eight values of z in place of four would be given by the formula. The proper limitation is imposed by the relation

$$\sqrt{p}\sqrt{q}\sqrt{r} = -\frac{G}{2},$$

which (lost sight of in squaring to obtain the value of pqr) requires such signs to be attached to each of the quantities $\sqrt{p}$, $\sqrt{q}$, $\sqrt{r}$, that their product may maintain the sign determined by the above equation; thus—

$$\sqrt{p}\sqrt{q}\sqrt{r} = \sqrt{p}(-\sqrt{q})(-\sqrt{r}) = (-\sqrt{p})\sqrt{q}(-\sqrt{r})$$
$$= (-\sqrt{p})(-\sqrt{q})\sqrt{r}$$

are all the possible combinations of $\sqrt{p}$, $\sqrt{q}$, $\sqrt{r}$ fulfilling this condition, provided that $\sqrt{p}$, $\sqrt{q}$, $\sqrt{r}$ retain the same signs throughout, whatever those signs may be. We may, however, remove all ambiguity as regards sign, and express in a single algebraic formula the four values of z, by eliminating one of the quantities $\sqrt{p}$, $\sqrt{q}$, $\sqrt{r}$ from the assumed value of z by means of the relation given above, and leaving the other two quantities unrestricted in sign. The expression for z becomes, therefore,

$$z = \sqrt{p} + \sqrt{q} - \frac{G}{2\sqrt{p}\sqrt{q}},$$

a formula free from all ambiguity, since it gives four, and only four, values of z when $\sqrt{p}$ and $\sqrt{q}$ receive their double signs: the sign given to each of these in the two first terms determining that which must be attached to it in the denominator of the third term. And

finally, restoring to p, q, and z their values given before, we have

$$ax+b=\sqrt{b^2-ac+a^2\theta_1}+\sqrt{b^2-ac+a^2\theta_2}-\frac{G}{2\sqrt{b^2-ac+a^2\theta_1}\sqrt{b^2-ac+a^2\theta_2}},$$

as the complete algebraic solution of the biquadratic equation ; θ_1 and θ_2 being roots of the equation

$$4a^3\theta^3-Ia\theta+J=0.$$

To assist the student in justifying Euler's apparently arbitrary assumption as to the form of solution of the biquadratic, we remark that, the second term of the equation in z being absent, the sum of the four roots is zero, or $z_1+z_2+z_3+z_4=0$; and consequently the functions $(z_1+z_2)^2$, etc., of which there are in general *six* (the combinations of four quantities two and two), are in this case reduced to *three* ; so that we may assume

$$(z_2+z_3)^2=(z_1+z_4)^2=4p,$$
$$(z_3+z_1)^2=(z_2+z_4)^2=4q,$$
$$(z_1+z_2)^2=(z_3+z_4)^2=4r\ ;$$

from which we have z_1, z_2, z_3, z_4, included in the formula

$$p+\sqrt{q}+\sqrt{r}.$$

We now proceed to express the roots of Euler's cubic (1), and also those of the reducing cubic (2), in terms of the roots α, β, γ, δ of the given biquadratic in x. Attending to the remarks above made with reference to the signs of the radicals, we may write the four values of $z\equiv ax+b$ as follows :—

$$\begin{aligned} a\alpha+b&=\ \ \sqrt{p}-\sqrt{q}-\sqrt{r}, \\ a\beta+b&=-\sqrt{p}+\sqrt{q}-\sqrt{r}, \\ a\gamma+b&=-\sqrt{p}-\sqrt{q}+\sqrt{r}, \\ a\delta+b&=\ \ \sqrt{p}+\sqrt{q}+\sqrt{r}\ ; \end{aligned} \qquad \ldots(3)$$

from which may be immediately derived the following expressions for p, q, r the roots of Euler's cubic :—

$$\begin{aligned} p&=\frac{a^2}{16}(\beta+\gamma-\alpha-\delta)^2, \\ q&=\frac{a^2}{16}(\gamma+\alpha-\beta-\delta)^2, \\ r&=\frac{a^2}{16}(\alpha+\beta-\gamma-\delta)^2. \end{aligned} \qquad \ldots(4)$$

Subtracting in pairs the equations (3), and making use of the relations above written between p, q, r and θ_1, θ_2, θ_3, we easily establish the following useful relations connecting the differences of the roots of the cubics (1) and (2) with the differences of the roots of the biquadratic :—

$$\begin{aligned} 4(q-r) &= 4a^2(\theta_2-\theta_3) = -a^2(\beta-\gamma)(\alpha-\delta), \\ 4(r-p) &= 4a^2(\theta_3-\theta_1) = -a^2(\gamma-\alpha)(\beta-\delta), \\ 4(p-q) &= 4a^2(\theta_1-\theta_2) = -a^2(\alpha-\beta)(\gamma-\delta). \end{aligned} \qquad \ldots(5)$$

Finally, from these equations, by aid of the relation $\theta_1+\theta_2+\theta_3=0$, we derive the values of θ_1, θ_2, θ_3 in terms of α, β, γ, δ, *viz.*,

$$\begin{aligned} 12\theta_1 &= (\gamma-\alpha)(\beta-\delta)-(\alpha-\beta)(\gamma-\delta), \\ 12\theta_2 &= (\alpha-\beta)(\gamma-\delta)-(\beta-\gamma)(\alpha-\delta), \\ 12\theta_3 &= (\beta-\gamma)(\alpha-\delta)-(\gamma-\alpha)(\beta-\delta). \end{aligned}$$

Examples

1. When the biquadratic has two equal roots, the reducing cubic has two equal roots, and conversely.

2. When the biquadratic has three roots equal, all the roots of the reducing cubic vanish, and consequently $I=0$, $J=0$.

3. When the biquadratic has two distinct pairs of equal roots, two of the roots of Euler's cubic vanish, and consequently $G=0$, $a^2I-12H^2=0$.

4. Prove the following relations between the biquadratic and Euler's cubic with respect to the nature of the roots :—

(*i*) When the roots of the biquadratic are all real, the roots of Euler's cubic are all real and positive.

(*ii*) When the roots of the biquadratic are all imaginary, the roots of Euler's cubic are all real, two being negative and one positive.

(*iii*) When the biquadratic has two real and two imaginary roots, Euler's cubic has two imaginary roots and one real positive root.

These results follow readily from equations (4) when the proper forms are substituted for α, β, γ, δ in the values of p, q, r. It is to be observed that all possible cases are here comprised, the biquadratic being supposed not to have equal roots. It follows that the converse of each of these propositions is true. Hence, when Euler's cubic has all its roots real and positive, we may conclude that all the roots of the biquadratic are real ; when Euler's cubic has negative roots, we conclude that all the roots of the biquadratic are imaginary ; and when Euler's cubic has imaginary roots, we conclude that the biquadratic has two real and two imaginary roots.

5. Prove that the roots of the biquadratic and the roots of the reducing cubic are connected by the following relations :—

(*i*) When the roots of the biquadratic are either all real, or all imaginary, the roots of the reducing cubic are all real ; and, conversely, when the roots of the reducing cubic are all real, the roots of the biquadratic are either all real or all imaginary.

(*ii*) When the biquadratic has two real, and two imaginary roots, the reducing cubic has two imaginary roots ; and, conversely, when the reducing cubic has two imaginary roots, the biquadratic has two real and two imaginary roots.

These results follow readily from the preceding example, since the roots of the two cubics (*i*) and (*ii*) are connected by a real linear relation.

6. When H is positive, the biquadratic has imaginary roots.

For in that case the roots of Euler's cubic cannot be all positive.

7. When I is negative, the biquadratic has two real and two imaginary roots. For the reducing cubic has in that case two imaginary roots (Ex. 12, p. 28).

8. When H and J are both positive, all the roots of the biquadratic are imaginary.

For since J is positive, the reducing cubic has a real negative root ; therefore, also Euler's cubic has a real negative root, since $t=a^2\theta-H$, and H is positive ; and this is case (*ii*) of Ex. 4. It is implied in this proof that the leading coefficient a is positive ; if $a\,J$ be substituted for J in the statement of the proposition no restriction as to the sign of a is necessary.

9. Show that the two biquadratic equations

$$A_0x^4+6A_2x^2\pm 4A_3x+A_4=0$$

have the same reducing cubic.

10. Find the reducing cubic of the two biquadratic equations

$$x^4-6lx^2\pm 8x\sqrt{l^3+m^3+n^3-3lmn}+3(4mn-l^2)=0.$$

[*Ans.* $\theta^3-3mn\theta-(m^3+n^3)=0$.

11. Prove that the eight roots of the equation

$$[x^4-6lx^2+3(4mn-l^2)]^2=64(l^3+m^3+n^3-3lmn)x^2$$

are given by the formula

$$\sqrt{l+m+n}+\sqrt{l+\omega m+\omega^2 n}+\sqrt{l+\omega^2 m+\omega n}.$$

(Compare Ex. 20, p. 27).

12. If the expression

$$\sqrt{l+m+n}+\sqrt{l+\omega m+\omega^2 n}+\sqrt{l+\omega^2 m+\omega n}$$

be a root of the equation

$$z^4+6Hz^2+4Gz+a^2I-3H^2=0,$$

determine H, I, J, in terms of l, m, n.

[*Ans.* $H=-l$, $a_0{}^2I=12mn$, $a_0{}^3J=-4(m^3+n^3)$.

13. Write down the formulas which express the root of a biquadratic in the particular cases when $I=0$, and $J=0$.

14. Express, by the aid of the reducing cubic, I and J in terms of the differences of the roots α, β, γ, δ. (*See* Exs. 16, 18, Art. 27.)

15. Express the product of the squares of the differences of the roots α, β, γ, δ in terms of I and J.

By means of the equations (5) above given, and the equation (2), p. 66, we obtain the result as follows :—

$$a^6(\beta-\gamma)^2(\gamma-\alpha)^2(\alpha-\beta)^2(\alpha-\delta)^2(\beta-\delta)^2(\gamma-\delta)^2=256(I^3-27J^2).$$

16. What is the quantity under the *final* square root (*viz.*, that which occurs under the cube root in the solution of the reducing cubic) in the formula expressing a root ? [*Ans.* $27J^2-I^3$.

17. Prove that the co-efficients of the equation of squared differences of the biquadratic equation $a_0x^4+4a_1x^3+6a_2x^2+4a_3x+a_4=0$ may be expressed in terms of a_0, H, I, J.

Removing the second term from the equation, we obtain

$$y^4+\frac{6H}{a_0^2}y^2+\frac{4G}{a_0^3}y+\frac{a_0^2I-3H^2}{a_0^4}=0;$$

and changing the signs of the roots, we have

$$y^4+\frac{6H}{a_0^2}y^2-\frac{4G}{a_0^3}\quad+\frac{a_0^2I-3H^2}{a_0^4}=0.$$

These transformations leave the functions $(\alpha-\beta)^2$, etc., unaltered ; but G becomes $-G$, the other co-efficients of the latter equation remaining unchanged ; therefore, G can enter the coefficients of the equation of squared differences in *even* powers only. And by aid of the identity of Art. 37, G^2 may be eliminated, introducing a_0, H, I, J. In a similar manner we may prove that every even function of the differences of the roots, α, β, γ, δ, may be expressed in terms of a_0, H, I, J, the function G of odd degree not entering.

62. Second Solution by Radicals of the Biquadratic. Let the biquadratic equation

$$ax^4+4bx^3+6cx^2+4dx+e=0$$

be put, as before, under the form

$$z^4+6Hz^2+4Gz+a^2I-3H^2=0,$$

where $z\equiv ax+b$.

We now assume as the general expression for a root of this equation

$$z=\sqrt{q}\sqrt{r}+\sqrt{r}\sqrt{p}+\sqrt{p}\sqrt{q},$$

a formula involving three independent radicals, $\sqrt{p}$, $\sqrt{q}$, $\sqrt{r}$.

Squaring twice, and reducing, we have

$$(z^2-qr-rp-pq)^2=4pqr(2z+p+q+r),$$

or

$$z^4-2(qr+rp+pq)z^2-8pqrz+(qr+rp+pq)^2-4(p+q+r)pqr=0.$$

Comparing this equation with the former equation in z, we easily find

$$qr+rp+pq=-3H,\ pqr=-\frac{G}{2},\ p+q+r=\frac{a^2I-12H^2}{2G};$$

whence, p, q, r are the roots of the equation

$$2Gt^3+(12H^2-a^2I)t^2-6HGt+G^2=0.$$

This equation may be readily transformed into Euler's cubic, or making directly the substitution

$$t=\frac{\frac{1}{2}G}{H-a^2\theta},$$

and putting for G^2 its value in terms of H, I, and J, we may reduce it to the standard form of the reducing cubic, *viz.*,

$$4a^3\theta^3-Ia\theta+J=0.$$

It is important to observe that in the present method of solution we meet with no ambiguity corresponding to that of Art. 61; for the expression here assumed as the value of z has, in virtue of the double signs of the radicals contained in it, *only four values*, while the form assumed for z in the preceding Article has eight values. This appears from the identical equation

$$2(\sqrt{q}\sqrt{r}+\sqrt{r}\sqrt{p}+\sqrt{p}\sqrt{q})\equiv(\sqrt{p}+\sqrt{q}+\sqrt{r})^2-p-q-r,$$

which shows that the number of distinct values of the radical expression of the present Article is the same as the number of values of $(\sqrt{p}+\sqrt{q}+\sqrt{r})^2$, namely four.

In order to express p, q, r in terms of the roots α, β, γ, δ of the biquadratic, we have, giving to x the four values α, β, γ, δ,

$$\begin{aligned}
z_1&\equiv a\alpha+b=\quad\sqrt{q}\sqrt{r}-\sqrt{r}\sqrt{p}-\sqrt{p}\sqrt{q},\\
z_2&\equiv a\beta+b=-\sqrt{q}\sqrt{r}+\sqrt{r}\sqrt{p}-\sqrt{p}\sqrt{q},\\
z_3&\equiv a\gamma+b=-\sqrt{q}\sqrt{r}-\sqrt{r}\sqrt{p}+\sqrt{p}\sqrt{q},\\
z_4&\equiv a\delta+b=\quad\sqrt{q}\sqrt{r}+\sqrt{r}\sqrt{p}+\sqrt{p}\sqrt{q}.
\end{aligned}$$

The student may easily satisfy himself that no combination of the signs of the radicals can lead to any value different from these four.

From the values of $z_2+z_3-z_1-z_4$, and $z_2z_3-z_1z_4$, we obtain

$$a(\beta+\gamma-\alpha-\delta)=-4\sqrt{q}\sqrt{r};$$
$$a^2(\beta\gamma-\alpha\delta)+ab(\beta+\gamma-\alpha-\delta)=4p\sqrt{q}\sqrt{r}.$$

From these and similar equations we have, employing the relation $G=-2pqr$, the following modes of expressing p, q, r in terms of the roots α, β, γ, δ:—

$$-p=a\frac{\beta\gamma-\alpha\delta}{\beta+\gamma-\alpha-\delta}+b=\frac{8G}{a^2(\beta+\gamma-\alpha-\delta)^2},$$

$$-q=a\frac{\gamma\alpha-\beta\delta}{\gamma+\alpha-\beta-\delta}+b=\frac{8G}{a^2(\gamma+\alpha-\beta-\delta)^2},$$

$$-r=a\frac{\alpha\beta-\gamma\delta}{\alpha+\beta-\gamma-\delta}+b=\frac{8G}{a^2(\alpha+\beta-\gamma-\delta)^2}.$$

63. Resolution of the Quartic into its Quadratic Factors. Let the quartic

$$ax^4+4bx^3+6cx^2+4dx+e$$

be supposed to be expressed as the difference of two squares* in the form

$$(ax^2+2bx+c+2a\theta)^2-(2Mx+N)^2.$$

Multiplying the given quartic by a, and comparing it with this expression, we have the following equations to determine M, N, and θ :—

$$M^2=b^2-ac+a^2\theta,\ MN=bc-ad+2ab\theta,\ N^2=(c+2a\theta)^2-ae.$$

Eliminating M and N from these equations, we find

$$4a^3\theta^3-(ae-4bd+3c^2)a\theta+ace+2bcd-ad^2-eb^2-c^3=0,$$

which is the reducing cubic before obtained.

From this equation, we have three values of $\theta(\theta_1,\ \theta_2,\ \theta_3)$, with three corresponding values of M^2, MN, N^2 ; and thus all the co-efficients of the assumed form for the quartic are determined in three distinct ways ; moreover, it should be noticed that to each value of M corresponds a *single* value of N, since

$$MN=bc-ad+2ab\theta.$$

The quartic

$$(ax^2+2bx+c+2a\theta)^2-(2Mx+N)^2$$

may plainly be resolved into the two quadratic factors

$$ax^2+2(b-M)x+c+2a\theta-N,$$
$$ax^2+2(b+M)x+c+2a\theta+N$$

When θ receives the three values θ_1, θ_2, θ_3, we obtain the three pairs of quadratic factors of the original quartic, and the problem is completely solved.

In order to make clear the connexion betweeen the present solution and the solution by radicals, let us suppose that the roots of the quadratic factors in the order above written are β, γ and α, δ ; and that the roots of the remaining pairs of quadratic factors are similarly γ, α and β, δ ; α, β and γ, δ. We have, therefore,

$$\beta+\gamma=-\frac{2}{a}(b-M_1),\ \gamma+\alpha=-\frac{2}{a}(b-M_2),\ \alpha+\beta=-\frac{2}{a}(b-M_3),$$
$$\alpha+\delta=-\frac{2}{a}(b+M_1),\ \beta+\delta=-\frac{2}{a}(b+M_2),\ \gamma+\delta=-\frac{2}{a}(b+M_3),$$

*The reduction of the quartic to the difference of two squares was the method first employed for the solution of the equation of the fourth degree. This mode of solution is due to *Ferrari*, although by some writers ascribed to *Simpson* (*see* Note A). The method explained in the following Article, in which the quartic is equated directly to the product of two quadratic factors, is due to *Descartes*.

where

$$M_1 \equiv \sqrt{b^2-ac+a^2\theta_1},\ M_2 \equiv \sqrt{b^2-ac+a^2\theta_2},\ M_3 \equiv \sqrt{b^2-ac+a^2\theta_3}.$$

Subtracting the last equations in pairs, we find

$$\beta+\gamma-\alpha-\delta=4\frac{M_1}{a},\ \gamma+\alpha-\beta-\delta=4\frac{M_2}{a},\ \alpha+\beta-\gamma-\delta=4\frac{M_3}{a}$$

and since

$$a+\beta+\gamma+\delta=-4$$

we obtain

$$\begin{aligned} a\alpha+b&=-M_1+M_2+M_3,\\ a\beta+b&=\ \ M_1-M_2+M_3,\\ a\gamma+b&=\ \ M_1+M_2-M_3,\\ a\delta+b&=-M_1-M_2-M_3. \end{aligned}$$

It appears, therefore, that the roots of the biquadratic are here expressed separately by formulas analogous to those of Art. 61. The values of M^2, *viz.*, $M_1{}^2$, $M_2{}^2$, $M_3{}^2$, are in fact identical with the roots of Euler's cubic in the preceding Article. There exists also with regard to the signs of the radicals involved in M_1, M_2, M_3, a restriction similar to that of Art. 61 ; since, in virtue of the assumptions above made with respect to the roots of the quadratic factors, we have the equation

$$a^2(\beta+\gamma-\alpha-\delta)(\gamma+\alpha-\beta-\delta)(\alpha+\beta-\gamma-\delta)=64M_1M_2M_3.$$

which implies the following relation (*see* Ex. 20, p. 41) :—

$$M_1M_2M_3=\tfrac{1}{2}G\ ;$$

and by means of this relation the signs of M_1, M_2, M_3, are restricted in the manner explained in the previous Article.

By aid of the equation last written we can eliminate M_3 from the expressions for the roots, and thus obtain, as in Article 61, all the roots of the biquadratic in a *single* formula, *viz.*,

$$ax+b=M_1+M_2-\frac{G}{2M_1M_2},$$

in which the radicals $M_1 \equiv \sqrt{b^2-ac+a^2\theta_1}$, and $M_2 \equiv \sqrt{b^2-ac+a^2\theta_2}$ are taken in complete generality.

Examples

1. Form the equation whose roots are λ, μ, ν, *viz.*,

$$\beta\gamma+\alpha\delta,\ \gamma\alpha+\beta\delta,\ \alpha\beta+\gamma\delta.$$

Adding the last co-efficients of the quadratic factors of the quartic, we have

$$\beta\gamma+\alpha\delta=4\theta_1+2$$

$$\gamma\alpha+\beta\delta=4\theta_2+2\,\frac{c}{a},$$

$$\alpha\beta+\gamma\delta=4\theta_3+2\,\frac{c}{a},$$

where θ_1, θ_2, θ_3 are the roots of the reducing cubic ; hence the required equation. (*Cf.* Exs. 4, 5, Art. 39). [*Ans.* $(ax-2c)^3-4I(ax-2c)+16J=0$.

2. Express, by means of the equations of the preceding example, the roots of the reducing cubic in terms of the roots of the biquadratic.

Substituting for $\frac{2c}{a}$ its value in terms of α, β, γ, δ, we find immediately

$$12\theta_1=2\lambda-\mu-\nu\equiv(\gamma-\alpha)(\beta-\delta)-(\alpha-\beta)(\gamma-\delta),$$
$$12\theta_2=2\mu-\nu-\lambda\equiv(\alpha-\beta)(\gamma-\delta)-(\beta-\gamma)(\alpha-\delta),$$
$$12\theta_3=2\nu-\lambda-\mu\equiv(\beta-\gamma)(\alpha-\delta)-(\gamma-\alpha)(\beta-\delta).$$

[*Cf* (6), Art. 61].

3. Verify, by means of the expression for θ_1, θ_2, θ_3 in Ex. 1, the conclusions of Ex. 5, Art. 61, with respect to the manner in which the roots of the biquadratic and reducing cubic are related.

4 Form the equation whose roots are the functions

$$\tfrac{1}{8}(\beta\gamma-\alpha\delta)(\beta+\gamma-\alpha-\delta),\ \tfrac{1}{8}(\gamma\alpha-\beta\delta)(\gamma+\alpha-\beta-\delta),\ \tfrac{1}{8}(\alpha\beta-\gamma\delta)(\alpha+\beta-\gamma-\delta).$$

From the quadratic factors of the quartic, we find

$$\frac{4M_1}{a}=\beta+\gamma-\alpha-\delta,\ -\frac{2N_1}{a}=\beta\gamma-\alpha\delta,$$

also

$$M_1N_1=bc-ad+2ab\theta_1=-a^2\phi_1,$$

the roots of the required cubic being represented by ϕ_1, ϕ_2, ϕ_3.

We obtain, therefore, the required equation by a linear transformation of the reducing cubic. [*Ans.* $(a^2\phi+bc-ad)^3-b^2I(a^2\phi+bc-ad)-2b^3J=0$.

5. Form the equation whose roots are

$$\frac{\beta\gamma-\alpha\delta}{\beta+\gamma-\alpha-\delta},\ \frac{\gamma\alpha-\beta\delta}{\gamma+\alpha-\beta-\delta},\ \frac{\alpha\beta-\gamma\delta}{\alpha+\beta-\gamma-\delta}.$$

If ϕ denote any one of these functions indifferently and θ the corresponding root of the reducing cubic, we have, employing former results,

$$-2\phi=\frac{MN}{M^2}=\frac{bc-ad+2ab\theta}{b^2-ac+a^2\theta};$$

and thus we obtain the required equation by a homographic transformation of the reducing cubic. This formula may be put under the more convenient form

$$a\phi+b=\frac{\frac{1}{2}G}{a^2\theta-H},$$

by means of which we obtain the required cubic in the following form :—

$$2G(a\phi+b)^3+(a^2I-12H^2)(a\phi+b)^2-6HG(a\phi+b)-G^2=0,$$

which, expanded and divided by a^3, becomes

$$2G\phi^3+(a^2e+6b^2c-9ac^2+2abd)\phi^2+2(abe+2b^2d-3acd)\phi+b^2e-ad^2=0.$$

(*Cf.* Ex. 14, p. 70).

6. Form the equation whose roots are

$$\frac{a^2}{4}(\beta\gamma-\alpha\delta)^2, \quad \frac{a^2}{4}(\gamma\alpha-\beta\delta)^2, \quad \frac{a^2}{4}(\alpha\beta-\gamma\delta)^2.$$

These are the three values of N^2 in the foregoing Article. Representing, as before, one of these values by ϕ, we find that the required equation may be obtained from the reducing cubic by means of the homographic transformation

$$\phi=\frac{2bcd-ad^2-eb^2+4abd\theta}{c-a\theta}.$$

7. Form the equation whose roots are

$$\frac{\beta\gamma-\alpha\delta}{(\beta+\gamma)\alpha\delta-(\alpha+\delta)\beta\gamma}, \quad \frac{\gamma\alpha-\beta\delta}{(\gamma+\alpha)\beta\delta-(\beta+\delta)\gamma\alpha}, \quad \frac{\alpha\beta-\gamma\delta}{(\alpha+\beta)\gamma\delta-(\gamma+\delta)\alpha\beta}.$$

The required equation is obtained from the reducing cubic by the homographic transformation

$$2\phi=\frac{cd-be+2ad\theta}{d^2-ce+ac\theta};$$

The result may be derived from Ex. 5 by changing the roots into their reciprocal, and making the corresponding changes in the coefficients.

64. The Resolution of the Quartic into Quadratic Factors. Second Method. Let the quartic

$$ax^4+4bx^3+6cx^2+4dx+e$$

be supposed to be resolved into the quadratic factors

$$a(x^2+2px+q)(x^2+2p'x+q').$$

We have, by comparing these two forms, the equations

$$p+p'=2\frac{b}{a}, \quad q+q'+4pp'=6\frac{c}{a}, \quad pq'+p'q=2\frac{d}{a}, \quad qq'=\frac{e}{a}. \quad \ldots(1)$$

If now we had any fifth equation of the form

$$F(p, q, p', q')=\phi,$$

we could eliminate p, p', q, q'; and thus find an equation giving the several values of ϕ.

The fifth equation might be assumed to be $pp'=\phi$, or $q+q'=\phi$; and in each case ϕ would be determined by a cubic equation, since each of these functions, when expressed in terms of the roots of the biquadratic, has three values only. It is more convenient, however, to assume

$$\phi=\frac{c}{a}-pp'=\frac{1}{4}\left(q+q'-\frac{2c}{a}\right),$$

the two functions of p, p', q, q' here involved being equal by the second of equations (1). We easily find, by the aid of those equations,

$$pq+p'q'=\frac{4abc-2a^2d}{a^3}+\frac{8b\phi}{a};$$

and eliminating p, p', q, q', by means of the identical relation

$$(p^2+p'^2)(q^2+q'^2) \equiv (pq'-p'q)^2+(pq+p'q')^2,$$

there results the equation

$$4a^3\phi^3 - Ia\phi + J = 0,$$

which is the reducing cubic obtained by the previous methods of solution.

Having thus found pp', or $q+q'$, we may complete the resolution of the quartic by means of the equations (1).

The reason for the assumption above made with regard to the form of the fifth equation is obvious. From a comparison of the assumed values of ϕ with the equations of Ex. 1, Art. 63, it appears that ϕ is the same as θ in the preceding Article ; and, therefore, we foresee that the elimination of $p, p'. q, q'$, must lead to an equation in ϕ identical with the reducing cubic before obtained. In general, if ϕ represent any function of the differences of λ, μ, ν, and consequently an *even* function of the differences of $\alpha, \beta, \gamma, \delta$ (*see* Ex. 18, Art. 27), the equation whose roots are the different values of ϕ cannot involve any functions of the coefficients except a, H, I, and J.

If ϕ be assumed equal to any of the expressions in the second of the following examples, the equation in φ whose roots are the different values of this expression is formed as in the above instance by the elimination of p, p', q, q'.

Examples

1. Resolve into quadratic factors

$$z^4+6Hz^2+4Gz+a^2I-3H^2.$$

Comparing this form with the product

$$(z^2+2pz+q)(z^2-2pz+q'),$$

we find the following equation for p :—

$$4p^6+12Hp^4+12\left(H^2-\frac{a^2I}{12}\right)p^2-G^2=0 ; \qquad [cf. (1), \text{Art. } 61]$$

and putting

$$a^2\varphi = p^2+H \equiv \tfrac{1}{4}(q+q'-2H),$$

this equation, when divided by a^3, becomes

$$4a^3\varphi^3 - Ia\varphi + J = 0.$$

2. If a quartic be resolved into the two quadratic factors

$$x^2+px+q, \; x^2+p'x+q',$$

prove that φ is determined by a cubic equation when it has all possible values corresponding to each of the following types :—

$$q+q', \; \frac{q-q'}{p-p'}, \; \frac{pq'-p'q}{p-p'} \cdot \frac{pq'-p'q}{q-q'} .$$

$$(p-p')^2, \; (p-p')(q-q'), \; (q-q')^2, \; (pq'-p'q)^2 ;$$

and by an equation of the sixth degree when it has all values corresponding to

$$p, q, p-p', q-q', pq'-p'q, \text{ or } p^2-4q.$$

Expressing these functions in terms of the roots, the number of possible values of each function becomes apparent.

65. Transformation of the Biquadratic into the Reciprocal Form. To effect this transformation we make the linear substitution $x=ky+\rho$ in the equation

$$ax^4+4bx^3+6cx^2+4dx+e=0,$$

which then assumes the form

$$ak^4y^4+4U_1k^3y^3+6U_2k^2y^2+4U_3ky+U_4=0,$$

where

$$U_1\equiv a\rho+b,\ U_2\equiv a\rho^2+2b\rho+c,\ U_3\equiv a\rho^3+3b\rho^2+3c\rho+d, \text{ etc.}$$

(*see* Art. 35). If this equation be reciprocal, we have two equations to determine k and ρ, *viz.*,

$$ak^4=U_4,\ k^3U_1=kU_3\ ;$$

eliminating k, we have the following equation for ρ :—

$$aU_3^2-U_1^2U_4=0\ ;$$

and since

$$k^2=\frac{U_3}{U_1}=\frac{a\rho^3+3b\rho^2+3c\rho+d}{a\rho+b},$$

there are two values of k, equal with opposite signs, corresponding to each value of ρ.

The equation

$$aU_3^2-U_1^2U_4=0,$$

when reduced by the substitutions (Arts. 36, 37)

$$a^2U_3\equiv U_1^3+3HU_1+G,$$
$$a^3U_4\equiv U_1^4+6HU_1^2+4GU_1+a^2I-3H^2,$$

becomes

$$2GU_1^3+(a^2I-12H^2)U_1^2-6GHU_1-G^2=0, \qquad \ldots(1)$$

which is a cubic equation determining $U_1\equiv a\rho+b$; and if we put

$$a\rho+b=\frac{\frac{1}{2}G}{a^2\theta-H},$$

θ is determined by the standard reducing cubic

$$4a^3\theta^3-Ia\theta+J=0.$$

This transformation* may be employed to solve the biquadratic; and it is important to observe that the cubic (1) which here presents

*This method of solving the biquadratic by transforming it to the reciprocal form was given by Mr. S.S. Greatheed in the *Camb. Math. Journ.* Vol. I.

itself differs from the cubic of Art. 62 only in having roots with contrary signs.

We proceed now to express k and ρ in terms of $\alpha, \beta, \gamma, \delta$, the roots of the biquadratic equation. Since the equation in y, obtained by putting $x=ky+\rho$, is reciprocal, its roots are of the form y_1, $y_2, \frac{1}{y_2}, \frac{1}{y_1}$; hence we may write

$$\alpha=ky_1+\rho,\ \beta=ky_2+\rho,\ \gamma=k\frac{1}{y_2}+\rho,\ \delta=k\frac{1}{y_1}+\rho;$$

and, therefore,

$$(\alpha-\rho)(\delta-\rho)=(\beta-\rho)(\gamma-\rho)=k^2,$$

from which we find

$$\rho=\frac{\beta\gamma-\alpha\delta}{\beta+\gamma-\alpha-\delta},$$

and

$$-k^2=\frac{(\gamma-\alpha)(\beta-\delta)(\alpha-\beta)(\gamma-\delta)}{(\beta+\gamma-\alpha-\delta)^2}.$$

An important geometrical interpretation may be given to the quantities k and ρ which enter into this transformation. Let the distances OA, OB, OC, OD, of four points A, B, C, D, on a right line from a fixed origin O on the line be determined by the roots $\alpha, \beta, \gamma, \delta$, of the equation

$$ax^4+4bx^3+6cx^2+4dx+e=0;$$

also let O_1, O_2, O_3 be the centres; and F_1, F_1'; F_2, F_2'; F_3, F_3', the foci of the three systems of involution determined by the three following pairs of quadratics :—

$$(x-\beta)(x-\gamma)=0,\ (x-\alpha)(x-\delta)=0;$$
$$(x-\gamma)(x-\alpha)=0,\ (x-\beta)(x-\delta)=0;$$
$$(x-\alpha)(x-\beta)=0.\ (x-\gamma)(x-\delta)=0.$$

We have then the equations

$$O_1B.O_1C=O_1A.O_1D=O_1F_1^2, \text{ etc.,}$$

which, transformed and compared with the equations

$$(\beta-\rho)(\gamma-\rho)=(\alpha-\rho)(\delta-\rho)=k^2, \text{ etc.,}$$

prove that the three values of ρ are OO_1, OO_2, OO_3, the distances of the three centres of involution from the fixed origin O. Also since $O_1F_1^2=k^2$, k has six values represented geometrically by the distances

$$O_1F_1,\ O_1F_1';\ O_2F_2,\ O_2F_2';\ O_3F_3,\ O_3F_3',$$

where $O_1F_1+O_1F_1'=0$, etc., as the distances are measured in opposite directions.

We can from geometrical considerations alone find the positions of the centres and foci of involution in terms of α, β, γ, δ, and thus confirm the results just established, as follows :—

Since the systems $\{F_1BF_1'C\}$ and $\{F_1AF_1'D\}$ are harmonic,

$$\frac{2}{F_1F_1'} = \frac{1}{F_1B} + \frac{1}{F_1C} = \frac{1}{F_1A} + \frac{1}{F_1D};$$

and if x represents the distance of F_1 or F_1' from the fixed origin O, we have

$$\frac{1}{x-\beta} + \frac{1}{x-\gamma} = \frac{1}{x-\alpha} + \frac{1}{x-\delta}$$

Solving this equation, we find

$$x = \frac{\beta\gamma-\alpha\delta}{\beta+\gamma-\alpha-\delta} \pm \frac{\sqrt{-(\gamma-\alpha)(\beta-\delta)(\alpha-\beta)(\gamma-\delta)}}{\beta+\gamma-\alpha-\delta}$$

or $\qquad x = \rho \pm k,$

whence $\qquad \rho = \dfrac{OF_1+OF_1'}{2}, \quad k = \pm\dfrac{OF_1-OF_1'}{2} = \pm O_1F_1.$

Example

Transform the cubic

$$ax^3+3bx^2+3cx+d$$

to the reciprocal form.

The assumption $x=ky+\rho$ leads to the equation

$$-GU_1^3+3H^2U_1^2+H^3=0, \quad \text{where } U_1 \equiv a\rho+b.$$

The values of ρ are easily seen to be

$$\frac{\beta\gamma-\alpha^2}{\beta+\gamma-2\alpha}, \quad \frac{\gamma\alpha-\beta^2}{\gamma+\alpha-2\beta}, \quad \frac{\alpha\beta-\gamma^2}{\alpha+\beta-2\gamma}.$$

The geometrical interpretation in this case is, that if three points A', B', C', be taken on the axis such that A' is the harmonic conjugate of A with respect to B and C, B' of B with respect to C and A, and C' of C with respect to A and B ; then we have the following values of ρ and k :—

$$\rho = \frac{OA+OA'}{2}, \quad -k = \frac{OA-OA'}{2}.$$

For the values of OA', OB', OC', in terms of α, β, γ, see Ex. 13, p. 71.

66. Solution of the Biquadratic by Symmetric Functions of the Roots. The possibility of reducing the solution of the biquadratic to that of a cubic by the present method depends on the possibility of forming functions of the four roots α, β, γ, δ, which admit of only three values when these roots are interchanged in every way. It will be seen on referring to Ex. 2, Art. 64, that several functions of this nature exist. These, like the analogous functions of Art. 59, possess an important property to be proved hereafter, *viz.*, any two such sets of three are so related that any one function of either

set is connected with some one function of the other set by a rational homographic relation in terms of the coefficients.

For the purposes of the present solution we employ the functions already referred to in Art. 55, since they lead in the most direct manner to the expressions for the roots of the biquadratic in terms of the coefficients. We proceed accordingly to form the equation whose roots are the three values of

$$t \equiv \left(\frac{\alpha+\theta\beta+\theta^2\gamma+\theta^3\delta}{4}\right)^2,$$

when the roots are interchanged in every way, and $\theta=-1$.

These values are

$$t_1 \equiv \left(\frac{\beta+\gamma-\alpha-\delta}{4}\right)^2,\quad t_2 \equiv \left(\frac{\gamma+\alpha-\beta-\delta}{4}\right)^2,\quad t_3 \equiv \left(\frac{\alpha+\beta-\gamma-\delta}{4}\right)^2;$$

and since

$$(\beta+\gamma-\alpha-\delta)^2 \equiv \Sigma\alpha^2+2\lambda-2\mu-2\nu,$$

$$\Sigma(\alpha-\beta)^2 \equiv 3\Sigma\alpha^2-2\lambda-2\mu-2\nu=-48\,H$$

we find the following values of t_1, t_2, t_3 :-

$$\frac{2\lambda-\mu-\nu}{12}-\frac{H}{a^2},\quad \frac{2\mu-\nu-\lambda}{12}-\frac{H}{a^2},\quad \frac{2\nu-\lambda-\mu}{12}-\frac{H}{a^2}$$

whence $$t_1+t_2+t_3=-3\frac{H}{a^2}$$

Again, since

$$\Sigma(2\mu-\nu-\lambda)(2\nu-\lambda-\mu)=-3(\lambda^2+\mu^2+\nu^2-\mu\nu-\nu\lambda-\lambda\mu)=-\frac{3}{2}\Sigma(\mu-\nu)^2,$$

and $$\Sigma(\mu-\nu)^2=24\frac{I}{a^2},$$

we have

$$t_2t_3+t_3t_1+t_1t_2=3\frac{H^2}{a^4}-\frac{1}{96}\Sigma(\mu-\nu)^2=\frac{3H^2}{a^4}-\frac{I}{4a^2};$$

also $$t_1t_2t_3=\frac{G^2}{4a^6}$$

Hence the equation whose roots are t_1, t_2, t_3 becomes

$$(a^2t)^3+3H(a^2t)^2+\left(3H^2-\frac{a^2I}{4}\right)(a^2t)-G^2=0;$$

or, substituting for G^2 its value from Art. 37,

$$4(a^2t+H)^3-a^2I(a^2t+H)+a^3J=0,$$

which is transformed into the standard reducing cubic by the substitution $a^2t+H=a^2\theta$.

To determine α, β, γ, δ we have the following equations :—

$$-\alpha+\beta+\gamma-\delta=4\sqrt{t_1},\ \alpha-\beta+\gamma-\delta=4\sqrt{t_2},\ \alpha+\beta-\gamma-\delta=4\sqrt{t_3},$$

along with
$$\alpha+\beta+\gamma+\delta=-4\frac{b}{a};$$

from which we find

$$\alpha=-\frac{b}{a}-\sqrt{t_1}+\sqrt{t_2}+\sqrt{t_3},$$

$$\beta=-\frac{b}{a}+\sqrt{t_1}-\sqrt{t_2}+\sqrt{t_3},$$

$$\gamma=-\frac{b}{a}+\sqrt{t_1}+\sqrt{t_2}-\sqrt{t_3},$$

$$\delta=-\frac{b}{a}-\sqrt{t_1}-\sqrt{t_2}-\sqrt{t_3}.$$

We have also from the above values of $\sqrt{t_1}$, $\sqrt{t_2}$ $\sqrt{t_3}$ the equation

$$\sqrt{t_1}\,\sqrt{t_2}\,\sqrt{t_3}=\frac{G}{2a^3},$$

by means of which one radical can be expressed in terms of the other two, and the general formula for a root shown to be the same as those previously given.

It is convenient, in connexion with the subject of this Article, to give some account of two functions of the roots of the biquadratic which possess properties analogous to those established in Art. 59 for corresponding functions of the roots of a cubic. Adopting a notation similar to that of the Article referred to, we may write these functions in terms of λ, μ, ν in the following form :—

$$L\equiv(\beta\gamma+\alpha\delta)+\omega(\gamma\alpha+\beta\delta)+\omega^2(\alpha\beta+\gamma\delta),$$
$$M\equiv(\beta\gamma+\alpha\delta)+\omega^2(\gamma\alpha+\beta\delta)+\omega(\alpha\beta+\gamma\delta).$$

By means of the equations of Ex. 1, Art. 63, these functions can be expressed in terms of the roots of the reducing cubic in the form

$$\tfrac{1}{4}L=\theta_1+\omega\theta_2+\omega^2\theta_3,\quad \tfrac{1}{4}M=\theta_1+\omega^2\theta_2+\omega\theta_3.$$

They may also be expressed, by aid of the equation of the present Article connecting t and θ, in terms of the values of t_1, t_2, t_3, as follows :— .

$$\tfrac{1}{4}L=t_1+\omega t_2+\omega^2t_3,\quad \tfrac{1}{4}M=t_1+\omega^2t_2+\omega t_3.$$

The functions L and M are as important in the theory of the biquadratic as the functions of Art. 59 in the theory of the cubic. The cubes of these expressions are the simplest functions of four

quantities which have but *two* values when these quantities are interchanged in every way; they are the roots of the reducing quadratic of the reducing cubic above written, and underlie every solution of the biquadratic which has been given.

Examples

1. Show that L and M are functions of the differences of α, β, γ, δ.

Increasing α, β, γ, δ by h, L and M remain unaltered, since $1+\omega+\omega^2=0$.

2. To find in terms of the coefficients the product of the squares of the differences of the roots α, β, γ, δ.

From the values of L and M in terms of θ_1, θ_2, θ_3, we find easily

$$\begin{aligned} 12\theta_1 &= L + M, & L - M &= (\beta-\gamma)(\alpha-\delta)(\omega^2-\omega), \\ 12\theta_2 &= \omega^2 L + \omega M, & \omega^2 L - \omega M &= (\gamma-\alpha)(\beta-\delta)(\omega^2-\omega), \\ 12\theta_3 &= \omega L + \omega^2 M, & \omega L - \omega^2 M &= (\alpha-\beta)(\gamma-\delta)(\omega^2-\omega). \end{aligned}$$

Again, from these equations, multiplying the terms on both sides together, and remembering that θ_1, θ_2, θ_3 are the roots of

$$4a^3\theta^3 - Ia\theta + J = 0,$$

we find

$$L^3 + M^3 = -432 \frac{J}{a^3},$$

$$L^3 - M^3 = 3\sqrt{-3}(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)(\alpha-\delta)(\beta-\delta)(\gamma-\delta);$$

also, adding the squares of the same terms, we have

$$2LM = 24 \frac{I}{a^2} = (\beta-\gamma)^2(\alpha-\delta)^2 + (\gamma-\alpha)^2(\beta-\delta)^2 + (\alpha-\beta)^2(\gamma-\delta)^2;$$

and, since

$$(L^3 - M^3)^2 \equiv (L^3 + M^3)^2 - 4L^3M^3,$$

substituting for the quantities their values derived from former equations, we have finally

$$a^6(\beta-\gamma)^2(\gamma-\alpha)^2(\alpha-\beta)^2(\alpha-\delta)^2(\beta-\delta)^2(\gamma-\delta)^2 = 256(I^3 - 27J^2).$$

3. Show by a comparison of the equations of Art. 59 with those of the present Article that the results of the former may be extended to the biquadratic by changing

$$\beta-\gamma,\ \gamma-\alpha,\ \alpha-\beta \text{ into } -(\beta-\gamma)(\alpha-\delta),\ -(\gamma-\alpha)(\beta-\delta),\ -(\alpha-\beta)(\gamma-\delta),$$

respectively; and, at the same time, H into $-\frac{4}{3} I$, and G into $16J$.

67. Equation of Squared Differences of a Biquadratic.

In a previous chapter (Art. 44) an account was given of the general problem of the formation of the equation of differences. It was proposed by Lagrange to employ this equation in practice for the purpose of separating the roots of a given numerical equation; and with a view to such application he calculated the general forms of the equation of squared differences in the cases of equations of the fourth and fifth degrees wanting the second term (see *Traite de la Resolution des Equations Numeriques*, 3rd ed., ch. v., and note III). Although for

practical purposes the methods of separation of the roots to be hereafter explained are to be preferred ; yet, in connexion with the subjects of the present chapter, the equation of squared differences of the biquadratic is of sufficient interest to be given here. We proceed accordingly to calculate this equation for a biquadratic written in the most general form. It will appear, in accordance with what was proved in Ex. 17, Art. 61, that the coefficients of the resulting equation can all be expressed in terms of a, H, I, and J.

The problem is equivalent to expressing the following product in terms of the coefficients of the biquadratic

$$[\phi-(\beta-\gamma)^2]\,[\phi-(\gamma-\alpha)^2]\,[\phi-(\alpha-\beta)^2]\,[\phi-(\alpha-\delta)^2]\,[\phi-(\beta-\delta)^2]\,[\phi-(\gamma-\delta)^2].$$

The most convenient mode of procedure is to group these six factors in pairs, and to express the three products (which we denote by π_1, π_2, π_3) separately in terms of the roots of the reducing cubic, and finally to express the product $\pi_1\pi_2\pi_3$ in terms of a, H, I, J.

$$\pi_1\equiv\phi^2-[(\beta-\gamma)^2+(\alpha-\delta)^2]\phi+(\beta-\gamma)^2(\alpha-\delta)^2\,;$$

and, by aid of the results of Art. 61 we easily derive the following expressions for $(\beta-\gamma)^2$, $(\alpha-\delta)^2$:—

$$4\left(\sqrt{\theta_2-\frac{H}{a^2}}-\sqrt{\theta_3-\frac{H}{a^2}}\right)^2,\quad 4\left(\sqrt{\theta_2-\frac{H}{a^2}}+\sqrt{\theta_3-\frac{H}{a^2}}\right)^2;$$

hence, without difficulty,

$$\pi_1\equiv\phi^2+\left(8\theta_1+16\,\frac{H}{a^2}\right)\phi+4\,\frac{I}{a^2}-48\theta_2\theta_3.$$

Introducing now for brevity the notation

$$16H\equiv a^2P,\ 4I\equiv a^2Q,\ 16J\equiv a^3R,\ \phi^2+P\phi+Q\equiv\psi,$$

π_1 becomes $\psi+8\theta_1\phi-48\theta_2\theta_3$.

Reducing the product $\pi_1\pi_2\pi_3$ by the result of Example 18, page 72, we obtain

$$\psi^3+3Q\psi^2-(4Q\phi^2+18R\phi)\psi-(8R\phi^3+12Q^2\phi^2+36QR\phi+27R^2)=0.$$

Finally, restoring the value of ψ, we have the equation of squared differences expressed in terms of P, Q, R, as follows :—

$$\phi^6+3P\phi^5+(3P^2+2Q)\phi^4+(P^3+8PQ-26R)\phi^3$$
$$+(6P^2Q-7Q^2-18PR)\phi^2+9Q(PQ-6R)\phi+4Q^3-27R^2=0.$$

The following is the final equation in terms of a, H, I, J* :—

$$a^6\phi^6+48a^4H\phi^5+8a^2(96H^2+a^2I)\phi^4+32(128H^3+16a^2HI-13a^3J)\phi^3$$
$$+16(384H^2I-7a^2I^2-288aHJ)\phi^2+1152(2HI-3aJ)I\phi+256(I^3-27J^2)=0.$$

It should be observed that the value above obtained for π_1 can be expressed as a quadratic function of θ_1 by aid of the equation $\theta_2\theta_3=\theta_1^2-\dfrac{I}{4a^2}$, and the subsequent calculation might have been conducted by eliminating θ_1 between this quadratic and the reducing cubic.

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* The equation of squared differences was first given in this form by Mr. M. Roberts in the *Nouvelles Annales de Mathematiques*, vol. xvi.

68. Criterion of the Nature of the Roots of the Biquadratic. Before proceeding with this investigation it is necessary to repeat what was before stated (Art. 43), that when any condition with respect to the nature of the roots of an algebraic equation is expressed by the sign of a function of the coefficients, these coefficients are supposed to represent real numerical quantities. It is assumed also, as in the Article referred to, that the leading coefficient does not vanish.

Using as before Δ to represent that function of the coefficients (called the *discriminant*) which is, when multiplied by a positive numerical factor, equal to the product of the squares of the differences of the roots, we have, from the results established in preceding Articles, the equation

$$a^6(\beta-\gamma)^2(\gamma-\alpha)^2(\alpha-\beta)^2(\alpha-\delta)^2(\beta-\delta)^2(\gamma-\delta)^2=256\,\Delta,$$

where $$\Delta \equiv I^3-27J^2.$$

It will be found convenient in what follows to arrange the discussion of the nature of the roots under three heads, according as—(1) Δ *vanishes*, or (2) *is negative*, or (3) *is positive*.

(1) *When Δ vanishes, the equation has equal roots.* This is evident from the value of Δ above written. Four distinct cases may be noticed—(α) *when two roots only are equal*, in which case I and J do not vanish separately ; (β) *when three roots are equal*, in which case $I=0$, and $J=0$, separately (*see* Ex. 2, Art. 61) ; (γ) *when two distinct pairs of roots are equal*, in which case we have the conditions $G=0$, $a^2I-12H^2=0$ (Ex. 3, Art. 61). It can be readily proved by means of the identity of Art. 37 that these conditions imply the equation $\Delta=0$; hence these two equations, along with the equation $\Delta=0$, are equivalent to two independent conditions only. Finally, we may have —(δ) *all the roots equal* ; in which case may be derived from Art. 61 the three independent conditions $H=0$, $I=0$, and $J=0$. These may be written in a form analogous to the corresponding conditions in case (4) of Art. 43.

(2) *When Δ is negative, the equation has two real and two imaginary roots.* This follows from the value of Δ in terms of the roots ; for when all the roots are real Δ is plainly positive ; and when the proper imaginary forms, *viz.*, $h\pm k\sqrt{-1}$, $h'\pm k'\sqrt{-1}$, are substituted for α, β, γ, δ, it readily appears that Δ is positive also when all the roots are imaginary.

(3) *When Δ is positive, the roots of the equation are either all real or all imaginary.* This follows also from the value of Δ, for we can

show by substituting for α, β the forms $h \pm k\sqrt{-1}$ that Δ is negative when two roots are real and two imaginary. In the case, therefore, when Δ is positive, this function of the coefficients is not by itself sufficient to determine completely the nature of the roots, for it remains still doubtful whether the roots are all real or all imaginary. The further conditions necessary to discriminate between these two cases may, however, be obtained from Euler's cubic (Art. 61) as follows :—In order that the roots of this cubic should be all real and positive, it is necessary that the signs should be alternately positive and negative ; and when the signs are of this nature the cubic cannot have a real negative root. We can, therefore, derive by the aid of Ex. 4 Art. 61, the following general conclusion applicable to this case :—*When Δ is positive the roots of the biquadratic are all imaginary in every case except when the following conditions are fulfilled, viz., H negative, and $a^2I - 12H^2$ negative ; in which case the roots are all real.*

Examples

1. Show that if H be positive, or if $H=0$ (and G not$=0$), the cubic will have a pair of imaginary roots.

2. Show that if H be negative, the cubic will have its roots —(1) all real and unequal, (2) two equal, or (3) two imaginary, according as G^2 is —(1) less than, (2) equal to, or (3) greater than $-4H^3$.

3. If the cubic equation

$$a_0x^3 + 3a_1x^2 + 3a_2x + a_3 = 0$$

have two roots equal to α ; prove

$$-\alpha = \frac{H_2}{H_1} = \frac{H_1}{H}$$

where $a_0a_2 - a_1^2 \equiv H$, $a_0a_3 - a_1a_2 \equiv 2H_1$, $a_1a_3 - a_2^2 \equiv H_2$.

4. If $$ax^3 + 3bx^2 + 3cx + d + k(x-r)^3$$

be a perfect cube, prove

$$(ac - b^2)r^2 + (ad - bc)r + (bd - c^2) = 0.$$

5. Find the condition that the cubic

$$ax^3 + 3bx^2 + 3cx + d$$

may be capable of being written under the form

$$l(x-\alpha_1)^3 + m(x-\beta_1)^3 + n(x-\gamma_1)^3,$$

where α_1, β_1, γ_1 are the roots of the cubic

$$a_1x^3 + 3b_1x^2 + 3c_1x + d_1 = 0.$$

Comparing the forms we have

$$a = l + m + n$$
$$-b = l\alpha_1 + m\beta_1 + n\gamma_1,$$
$$c = l\alpha_1^2 + m\beta_1^2 + n\gamma_1^2,$$
$$-d = l\alpha_1^3 + m\beta_1^3 + n\gamma_1^3.$$

Also $$a_1\alpha_1^3+3b_1\alpha_1^2+3c_1\alpha_1+d_1=0, \text{ etc.}$$

Whence, multiplying these equations by d_1, $3c_1$, $3b_1$, a_1, respectively, and adding, we find the required condition

$$(ad_1-a_1d)-3(bc_1-b_1c)=0.$$

6. If α, β, γ be the roots of the cubic equation

$$a_0x^3+3a_1x^2+3a_2x+a_3=0\,;$$

rationalize the equation

$$\sqrt[4]{x-\alpha}+\sqrt[4]{x-\beta}+\sqrt[4]{x-\gamma}=0\,;$$

and express the result in terms of coefficients a_0, a_1, a_2, a_3.

[*Ans.* $125U_1^4+360HU_1^2+128GU_1-48H^3=0$.

7. If α_1, β_1 and α_2, β_2 be the roots of the quadratic equations

$$a_1x^2+2b_1x+c_1=0,\ a_2x^2+2b_2x+c_2=0\,;$$

find the equation whose roots are the four values of $\alpha_1\alpha_2$.

Let $$H_1\equiv a_1c_1-b_1^2,\ H_2\equiv a_2c_2-b_2^2.$$

[*Ans.* $(a_1a_2\varphi^2-2b_1b_2\varphi+c_1c_2)^2-4H_1H_2\varphi^2=0$.

N.B.—This and the two following Examples may be solved by expressing by radicals involving the coefficients.

8. Employing the notation of Ex. 7, form the equation whose roots are the four values of $\dfrac{\alpha_1+\alpha_2}{2}$.

Let $$2K_{12}\equiv a_1c_2+a_2c_1+2b_1b_2.$$

[*Ans.* $[2a_1a_2\varphi^2+2(a_1b_2+a_2b_1)\varphi+K_{12}]^2-H_1H_2=0$.

In this Example the resulting biquadratic is such that $G=0$.

9. In the same case, if $\varphi=\frac{1}{2}(\alpha_1-\alpha_2)^2$, form the equation whose roots are the several values of φ.

Let $$M\equiv a_1b_2-a_2b_1,\ 2H_{12}\equiv a_1c_2+a_2c_1-2b_1b_2.$$

[*Ans.* $[(2a_1a_2\varphi+H_{12})^2-2M^2\varphi+H_1H_2]^2=4H_1H_2(a_1a_2\varphi+H_{12})^2$.

10. Show that when the biquadratic has a double root, the cubic whose roots are the values of ρ (Art. 65) has the *same* double root; and find what this cubic becomes when the biquadratic has three roots equal.

11. If H and J be both positive, prove directly (without the aid of Euler's cubic) that the roots of the biquadratic are all imaginary.

It appears from the expression for H in terms of the roots (Ex. 19, p. 42) that when H is positive there must be at least one pair of imaginary roots $h\pm k\sqrt{-1}$. Now diminishing all the roots by h, and dividing them by k (which transformations will not alter the character of the other pair of roots γ, δ, nor the signs of H and J), the biquadratic may be put under the form

$$(x^2+4px+q)(x^2+1),$$

or $$x^4+4px^3+6cx^2+4px+q, \text{ where } 6c=q+1\,;$$

whence $$H=c-p^2,\ I=q-4p^2+3c^2,$$

$$J=qc+2p^2c-p^2(q+1)-c^3=c(q-4p^2-c^2),$$

and, therefore,

$$q-4p^2=c^2+\frac{J}{c}=(H+p^2)^2+\frac{J}{H+p^2},$$

or
$$-\left(\frac{\gamma-\delta}{2k}\right)^2=(H+p^2)^2+\frac{J}{H+p^2},$$
proving that γ and δ are imaginary when H and J are positive (*cf*. Ex. 8. Art. 61).

12. If the biquadratic has two distinct pairs of equal roots, prove directly the relations
$$a_0^2I=12H^2,\ a_0^3J=8H^3.$$
In this case the biquadratic divided by a_0 assumes the form
$$(x-\alpha)^2(x-\beta)^2\equiv\left\{\left(x-\frac{\alpha+\beta}{2}\right)^2-\left(\frac{\alpha-\beta}{2}\right)^2\right\}^2=\left(\frac{z^2-k^2}{a_0^2}\right)^2,$$
where
$$z=a_0x+a_1,\ \text{and}\ \frac{k}{a_0}=\frac{\alpha-\beta}{2};$$
whence, comparing the forms
$$z^4-2k^2z^2+k^4$$
and
$$z^4+6Hz^2+4Gz+a_0^2I-3H^2,$$
we find
$$3H=-k^2,\ G=0,\ a_0^2I-3H^2=k^4,$$
from which the above relations immediately follow. The student will easily establish the identity of these relations with those of Ex. 3. Art. 61. Also it should be noticed that in this case only one square root is involved in the solution of the biquadratic [coming from the solution of the biquadratic $(x-\alpha)(x-\beta)$].

13. Find the condition that the biquadratic may be capable of being put under the form
$$l(x^2+2px+q)^2+m(x^2+2px+q)+n.$$
In this case the second and fourth co-efficients are removed by the same transformation, and the general solution involves only two square roots.]

[*Ans*. $G=0$.

14. Prove that J vanishes for the biquadratic
$$m(x-n)^4-n(x-m)^4.$$

15. If the roots of a biquadratic, α, β, γ, δ represent the distances of four points from an origin on a right line; prove that when these points form a harmonic division on the line the roots of Euler's cubic are in arithmetic progression, and the roots of the Art. 62 in harmonic progression.

16. Form the equation whose roots are the six anharmonic functions of four points in a right line determined by the equation
$$a_0x^4+4a_1x^3+6a_2x^2+4a_3x+a_4=0.$$
The six anharmonic ratios are
$$\varphi_1,\ \frac{1}{\varphi_1},\ \varphi_2,\ \frac{1}{\varphi_2},\ \varphi_3,\ \frac{1}{\varphi_3},$$
where
$$\varphi_1=-\frac{(\alpha-\beta)(\gamma-\delta)}{(\gamma-\alpha)(\beta-\delta)}=\frac{\lambda-\mu}{\lambda-\nu}=\frac{\theta_1-\theta_2}{\theta_1-\theta_3},$$
$$\varphi_2=-\frac{(\beta-\gamma)(\alpha-\delta)}{(\alpha-\beta)(\gamma-\delta)}=\frac{\mu-\nu}{\mu-\lambda}=\frac{\theta_2-\theta_3}{\theta_2-\theta_1},$$
$$\varphi_3=-\frac{(\gamma-\alpha)(\beta-\delta)}{(\beta-\gamma)(\alpha-\delta)}=\frac{\nu-\lambda}{\nu-\mu}=\frac{\theta_3-\theta_1}{\theta_3-\theta_2};$$
also the equation whose roots are
$$(\beta-\gamma)(\alpha-\delta),\ (\gamma-\alpha)(\beta-\delta),\ (\alpha-\beta)(\gamma-\delta)$$

is one of the cubics

$$a_0^2t^3-12a_0It\pm 16\sqrt{I^3-27J^2}=0.$$

The equation whose roots are the ratios, with sign changed, of the roots of *either* of these cubics is

$$4\Delta(\varphi^2-\varphi+1)^3-27I^3\varphi^2(\varphi-1)^2=0 \qquad (\textit{see}\ \text{Ex. 15, p. 72}),$$

where

$$\Delta\equiv I^3-27J^2.$$

The roots of the equation in φ are the six anharmonic ratios. This equation can be written in a more expressive form, as will appear from the following propositions :—

(*a*) The six anharmonic ratios may be expressed in terms of any one of them, as follows :—

$$\varphi,\ \frac{1}{\varphi},\ 1-\varphi,\ \frac{1}{1-\varphi},\ \frac{\varphi-1}{\varphi},\ \frac{\varphi}{\varphi-1}.$$

From the identical equation

$$(\beta-\gamma)(\alpha-\delta)+(\gamma-\alpha)(\beta-\delta)+(\alpha-\beta)(\gamma-\delta)\equiv 0$$

we have the relations

$$\varphi_1+\frac{1}{\varphi_3}=1,\ \ \varphi_2+\frac{1}{\varphi_1}=1,\ \varphi_3+\frac{1}{\varphi_2}=1,$$

which determines all the anharmonic ratios in terms of any one of them.

(*b*) If two of the anharmonic ratios become equal, the six values of φ are $-\omega$ and $-\omega^2$, each occurring three times ; and in this case $I=0$.

For suppose $\varphi_1=\varphi_2$; we have then from the second of the above relations

$$\varphi_1^2-\varphi_1+1=0,$$

whence

$$\varphi_1=-\omega,\ \text{or}\ -\omega^2\ ;$$

and substituting either of these values for φ in (*a*), we find all the anharmonic ratios.

Also since

$$\frac{\lambda-\mu}{\lambda-\nu}+\frac{\mu-\nu}{\lambda-\mu}=0,\ \text{or}\ \Sigma(\mu-\nu)^2=0,$$

we have

$$I\equiv a_0a_4-4a_1a_3+3a_2^2=0.$$

(*c*) When one of the ratios is harmonic, the six values of φ are -1, 2, $\frac{1}{2}$, each occurring twice ; and in this case $J=0$; for if

$$\varphi_1=-1,\ \frac{\lambda-\mu}{\lambda-\nu}=-1,\ \text{or}\ 2\lambda-\mu-\nu=0,$$

one of the factors of J (*see* Ex. 18, p. 41).

(*d*) These results, as well as the converse propositions, may be proved by writing the sextic in φ under the following form (*see* Ex. 12, p. 97).

$$I^3[(\varphi+1)(\varphi-2)(\varphi-\tfrac{1}{2})]^2=27J^2[(\varphi+\omega)(\varphi+\omega^2)]^3$$

17. Show that the equation

$$\left(\frac{x^2+14x+1}{\rho^4+14\rho^2+1}\right)^3=\frac{x(x-1)^4}{\rho^2(\rho^2-1)^4}$$

is satisfied by the solutions which follow :

$$\rho^2,\ \frac{1}{\rho^2},\ \left(\frac{1+\theta\sqrt{\rho}}{1-\theta\sqrt{\rho}}\right)^4,\ \text{where}\ \theta^4=1.$$

18. Express $\Sigma(\alpha-\beta)^4(\gamma-\delta)^2$ as a rational function of $\theta_1, \theta_2, \theta_3$; and ultimately in terms of the co-efficients of the quartic.

$$\left[Ans. \quad -128\Sigma(\theta_2-\theta_3)^2\left(\theta_1+\frac{2H}{a^2}\right)=-\frac{96}{a^4}(4HI+3aJ).\right.$$

19. Express

$$(\beta^2-\gamma^2)^2(\alpha^2-\delta^2)^2+(\gamma^2-\alpha^2)^2(\beta^2-\delta^2)^2+(\alpha^2-\beta^2)^2(\gamma^2-\delta^2)^2$$

as a rational function of $\theta_1, \theta_2, \theta_3$.

This symmetric function is equivalent to

$$(\mu^2-\nu^2)^2+(\nu^2-\lambda^2)^2+(\lambda^2-\mu^2)^2=256\Sigma(\theta_2-\theta_3)^2\left(\theta_1-\frac{c}{a}\right)^2.$$

20. Form the equation whose roots are the several products in pairs of the roots of a biquadratic.

The required equation is the product of three factors of the type

$$(\phi-\beta\gamma)(\phi-\alpha\delta)=\phi^2-\lambda\phi+\frac{e}{a}=\phi^2-2\frac{c}{a}\phi+\frac{e}{a}-4\phi\theta_1.$$

[*Ans.* $(a\phi^2-2c\phi+e)^3-4I\phi^2(a\phi^2-2c\phi+e)+16J\phi^3=0.$

21. Form the equation whose roots are the several values of $\frac{\alpha+\beta}{2}$, where $\alpha, \beta, \gamma, \delta$ are the roots of a biquadratic.

The required equation is the product of three factors of the type

$$\left(\phi-\frac{\beta+\gamma}{2}\right)\left(\phi-\frac{\alpha+\delta}{2}\right)=\phi^2+2\frac{b}{a}\phi+\frac{\mu+\nu}{4}=\phi^2+2\frac{b}{a}\phi+\frac{c}{a}-\theta_1.$$

[*Ans.* $4(a\phi^2+2b\phi+c)^3-I(a\phi^2+2b\phi+c)+J=0.$

22. Prove

$$\Sigma\frac{1}{(\alpha-\beta)^2}=\frac{9I}{2}\left(\frac{3aJ-2HI}{I^3-27J^2}\right).$$

From the expressions for $\alpha, \beta, \gamma, \delta$ in terms of $\theta_1, \theta_2, \theta_3$, we have

$$\Sigma\frac{1}{(\alpha-\beta)^2}=-\frac{1}{2a^2}\left\{\frac{a^2\theta_1+2H}{(\theta_2-\theta_3)^2}+\frac{a^2\theta_2+2H}{(\theta_3-\theta_1)^2}+\frac{a^2\theta_3+2H}{(\theta_1-\theta_2)^2}\right\},$$

which may be expressed in terms of a, H, I, J, as above.

23. Prove
$$\Sigma\frac{\theta_1^m}{(\theta_2-\theta_3)^2}=0,$$

if $I=0$, and m of the form $3p$ or $3p+1$, p being a positive integer.

24. Prove that

$$U\equiv ax^2+cy^2+ez^2+2dyz+2czx+2bxy$$

can be resolved into the sum of difference of two squares if

$$J\equiv ace+2bcd-ad^2-eb^2-c^3=0.$$

Here $\quad aU\equiv(ax+by+cz)^2+(ac-b^2)y^2+2(ad-bc)yz+(ae-c^2)z^2.$

and $\quad (ac-b^2)y^2+2(ad-bc)yz+(ae-c^2)z^2$

is a perfect square if

$$(ac-b^2)(ae-c^2)=(ad-bc)^2,$$

or $J=0$.

25. If $\alpha, \beta, \gamma, \delta$ be the roots of the equation

$$a_0x^4+4a_1x^3+6a_2x^2+4a_3x+a_4=0,$$

solve, in terms of the co-efficients a_0, a_1, etc., the equation

$$\sqrt{x-\alpha}+\sqrt{x-\beta}+\sqrt{x-\gamma}+\sqrt{x-\delta}=0.$$

When $$\sqrt{\alpha}+\sqrt{\beta}+\sqrt{\gamma}+\sqrt{\delta}=0$$

is rationalized, and the co-efficients substituted for α, β, γ, δ, we have

$$(3a_0a_2-2a_1^2)^2=a_0^3a_4.$$

Now, substituting U_0, U_1, U_2, U_3, U_4 for a_0, a_1, a_2, a_3, a_4, and reducing, we find

$$a_0x+a_1=\frac{1}{G}\left(3H^2-\frac{a_0^2I}{4}\right).$$

26. To obtain the equation of differences of a biquadratic, the equation of semi-sums (Ex. 21, p. 123), and to solve the biquadratic by one and the same transformation.

Substituting $x'+\rho$ for x, and using the notation of Art. 65, we have

$$ax'^4+4U_1x'^3+6U_2x'^2+4U_3x'+U_4=0.$$

We can suppose x' and ρ to have such values as satisfy the two equations

$$ax'^4+6U_2x'^2+U_4=0,\ U_1x'^2+U_3=0\ ;$$

from which it appears that corresponding to any value of ρ there are two values of x' equal with opposite signs, and when x'^2 is eliminated we find an equation of the sixth degree for ρ. To obtain the values of ρ and x' in terms of the roots α, β, γ, δ, of the biquadratic, assume

$$\rho_1+x'_1=\alpha,\ \rho_1-x_1'=\beta,\ \text{whence } 2\rho_1=\alpha+\beta,\ 2x_1'=\alpha-\beta.$$

The equation in x', therefore, obtained by eliminating, ρ, is the equation of semi-differences, and the sextic in ρ the equation of semi-sums. By the mode of reduction of Art. 65 the latter equation can be readily expressed in the form

$$4U_2^3-IU_2+J=0. \qquad \text{(Compare Ex. 21.)}$$

To solve the biquadratic, we have from the last equation $U_2=a\theta$, where θ is a root of the reducing cubic ; hence

$$U_1=a\rho+b=\sqrt{a^2\theta-H}\ ;\ x'^2=\frac{-U_3}{U_1}=\frac{-1}{a^2}\left(U_1^2+3H+\frac{G}{U_1}\right);$$

from which, finally,

$$ax+b=U_1+ax'=\sqrt{a^2\theta-H}+\sqrt{-a^2\theta-2H-\frac{G}{\sqrt{a^2\theta-H}}}:$$

an expression with only four values, in which the root of the biquadratic is expressed in terms of a single root of the reducing cubic.

27. Prove that every rational algebraic function of a root θ of a given cubic equation can in general be reduced to the form

$$\frac{C_0+C_1\theta}{D_0+D_1\theta}.$$

Let the given function be $\dfrac{\phi(\theta)}{\psi(\theta)}$, where $\phi(\theta)$ and $\psi(\theta)$ are rational integral functions of θ of any degree. By successive substitutions from the given cubic each of these may be reduced to a quadratic. Hence the given function is reducible to the form

$$\frac{c_0+c_1\theta+c_2\theta^2}{d_0+d_1\theta+d_2\theta^2}.$$

Equating this to the form written above, and reducing by the given cubic, we obtain an identical equation, *viz.*, $L_0+L_1\theta+L_2\theta^2\equiv 0$, where L_0, L_1, L_2 are linear functions of C_0, C_1, D_0, D_1. We have, therefore, the three equations $L_0=0$, $L_1=0$, $L_2=0$, to determine the ratios of C_0, C_1, D_0, D_1.

28. Prove that the solution of the biquadratic does not involve the extraction of a cube root when any relation among the roots α, β, γ, δ exists which can be expressed by the vanishing of a rational function of a root θ of the reducing cubic.

Any rational function of θ can always be depressed to the second degree, as in the preceding example. Hence the determination of θ will not involve the extraction of a cube root; and the formula of Ex. 26 shows that the expression for the root of the biquadratic will not then involve any cube root.

29. Find in each case the relation which connects the roots of the biquadratic when the equation

$$4\rho^3-I\rho+J=0$$

is satisfied by any of the following values of ρ :—

$$(1)\ \frac{H}{a},\ (2)\ c,\ (3)\ 0,\ (4)\ -\frac{\sqrt{ac-c}}{2},\ (5)\ \sqrt[3]{-\frac{J}{4}},\ (6)\ \sqrt{\frac{I}{12}},\ (7)\ \frac{3J}{2I},\ (8)\ -\frac{ad-bc}{2b}.$$

[*Ans.* (1) $\beta+\gamma-\alpha-\delta=0$, (2) $\beta+\gamma=0$, (3) $(\gamma-\alpha)(\beta-\delta)-(\alpha-\beta)(\gamma-\delta)=0$, (4), (8) $\beta\gamma-\alpha\delta=0$, (5) $(\gamma-\alpha)(\beta-\delta)-\omega(\alpha-\beta)(\gamma-\delta)=0$, (6), (7) $(\beta-\gamma)=0$.

30. Prove the identity

$$a_0^6(I^3-27J^2)\equiv(a_0^2I-3H^2)(a_0^2I-12H^2)^2+27G^2(G^2+2a_0^3J).$$

This may be proved as follows :—Putting $a_1=0$ in the values of I and J, and expanding, it readily appears that the part of Δ independent of a_1 may be thrown into the form

$$a_0a_4(a_0a_4-9a_2^2)^2+27a_0a_3^2(2a_0a_2a_4-a_0a_3^2-2a_2^3).$$

Now, replacing a_2, a_3, a_4 by A_2, A_3, A_4, and substituting for the latter quantities the values of Art. 37, we obtain the result. —Mr. *M. Roberts.*

31. When a biquadratic has two equal roots, prove that Euler's cubic has two equal roots whose common value is

$$\frac{3aJ-2HI}{2I};$$

and hence show that the remaining two roots of the biquadratic in this case are real, equal, or imaginary, according as $2HI-3aJ$ is negative, zero, or positive.

32. Prove that when a biquadratic has —(1) two distinct pairs of equal roots the last two roots of the equation of squared differences (Art. 67) vanish, giving the conditions $\Delta=0$, $2HI-3aJ=0$; and when it has —(2) three roots equal, the last three terms of this equation vanish, giving the conditions $I=0$, $J=0$; and show the equivalence of the conditions in the former case with those already obtained in Ex. 3 Art. 61, and Ex. 12, p. 121. Prove also that the equation of squared differences reduces in the former case to $\varphi^2(a^2\varphi+12H)^4$, and in the latter case to $\varphi^3(a^2\varphi+16H)^3$.

CHAPTER VII

PROPERTIES OF THE DERIVED FUNCTIONS

69. Graphic Representation of the Derived Function. Let APB be the curve representing the polynomial $f(x)$, and P the point on it corresponding to any value of the variable $x=OM$. We proceed to determine the mode of representing the value of $f'(x)$ at the point P. Take a second point Q on the curve, corresponding to a value of x which exceeds OM by a small quantity h. Thus $OM=x$, $MN=h$, $ON=x+h$; also $PM=f(x)$, $QN=f(x+h)$.

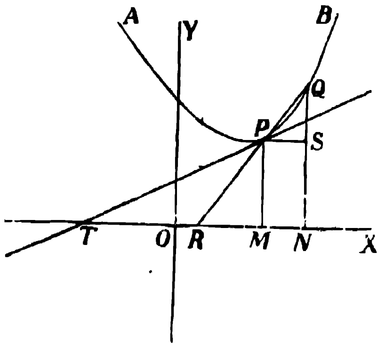

Fig. 5.

The expansion of Art. 6 gives

$$f(x+h)=f(x)+f'(x)h+\frac{f''(x)}{1.2}h^2+\cdots$$

or
$$\frac{f(x+h)-f(x)}{h}=f'(x)+\frac{f''(x)}{1.2}h+\cdots \qquad \ldots(1)$$

But
$$\frac{f(x+h)-f(x)}{h}=\frac{QS}{MN}=\frac{QS}{PS}=\tan QPS=\tan PRN.$$

Now, when h is indefinitely diminished, the point Q approaches, and ultimately coincides with, P; the chord PQ becomes the tangent PT to the curve at P; the angle PRN becomes PTM. Also all terms of the right-hand member of equation (1) except the first diminish indefinitely, and ultimately vanish when $h=0$. The equation (1) becomes, therefore,

$$\tan PTM=f'(x);$$

from which we conclude that *the value assumed by the derived function $f'(x)$ on the substitution of any value of x is represented by the tangent of the angle made with the axis OX by the tangent at the corresponding point to the curve representing the function $f(x)$.*

70. Maximum and Minimum Values of a Polynomial.

Theorem. *Any value of x which renders $f(x)$ a maximum or minimum is a root of the derived equation $f'(x)=0$.*

Let α be a value of x which renders $f(x)$ a *minimum*. We proceed to prove that $f'(x)=0$. Let h represent a small increment or decrement of x. We have, since $f(\alpha)$ is a minimum,

$$f(\alpha)<f(\alpha+h), \text{ also } f(\alpha)<f(\alpha-h);$$

hence $f(\alpha+h)-f(\alpha)$, and $f(\alpha-h)-f(\alpha)$ are both positive, *i.e.*, the following two expressions are positive :—

$$f'(\alpha)h+\frac{f''(\alpha)}{1.2}h^2\ldots$$

$$-f'(\alpha)h+\frac{f''(\alpha)}{1.2}h^2-\ldots$$

Now, when h is very small, we know (Art. 5) that the signs of these expressions are the same as the signs of their first terms; hence, in order that both should be positive, $f'(\alpha)$ must vanish; and, moreover, $f''(\alpha)$ *must be positive*. An exactly similar proof shows that when $f(\alpha)$ is a *maximum* $f'(\alpha)=0$, *and* $f''(\alpha)$ *is negative*. Thus in order to find the maximum and minimum values of a polynomial $f(x)$, we must solve the equation $f'(x)=0$, and substitute the roots in $f(x)$. Each root will furnish a maximum or minimum, the criterion to decide between these being the sign of $f''(x)$ when the root is substituted in it—*when* $f''(x)$ *is negative, the value is a maximum; and when* $f''(x)$ *is positive, the value is a minimum.*

The theorem of this Article follows at once from the construction of Art. 69; for it is plain that when the value of $f(x)$ is a maximum, as at P, P' (Fig. 6), or a minimum, as at p, p', the tangent to the curve will be parallel to the axis OX, and, consequently,

$$\tan PTM=f'(x)=0.$$

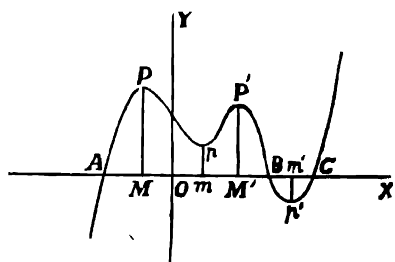

Fig. 6.

Fig. 6 represents a polynomial of the 5th degree. Corresponding to the four roots of $f'(x)=0$ (supposed all real in this case), *viz.*, OM, Om, OM', Om', there are two maxima, MP, $M'P'$; and two minima, mp, $m'p'$.

Examples

1. Find the max. or min. value of

$$f(x)\equiv 2x^2+x-6.$$
$$f'(x)=4x+1,\ f''(x)=4.$$

$$x=-\frac{1}{4} \text{ makes } f(x)=\frac{-49}{8} \text{ a minimum.}$$

(*See* Fig. 2, p. 12.)

2. Find the max. and min. values of

$$f(x) \equiv 2x^3 - 3x^2 - 36x + 14,$$
$$f'(x) = 6(x^2 - x - 6), f''(x) = 6(2x - 1).$$
$x = -2$ makes $f(x) = 68$, a maximum.
$x = \ \ 3$ makes $f(x) = -67$, a minimum.

3. Find the max. and min. values of

$$f(x) \equiv 3x^4 - 16x^3 + 6x^2 - 48x + 7.$$

Here $f'(x) = 0$ has only one real root, $x = 4$; and it gives a minimum value, $f(x) = -345$.

4. Find the max. and min. values of

$$f(x) \equiv 10x^3 - 17x^2 + x + 6.$$

The roots of $f'(x)$ are, approximately, $\cdot 0302$, $1 \cdot 1031$. The former gives a maximum value, the latter a minimum. [*See* Fig. 3, p. 12].

71. Rolle's Theorem. *Between two consecutive real roots a and b of the equation $f(x) = 0$ there lies at least one real root of the equation $f'(x) = 0$.*

For as x increases from a to b, $f(x)$, varying continuously from $f(a)$ to $f(b)$, must begin by increasing and then diminish, or must begin by diminishing and then increase. It must, therefore, pass through at least one maximum or minimum value during the passage from $f(a)$ to $f(b)$. This value ($f(\alpha)$, suppose) corresponds to some value α of x between a and b, which by the theorem of Art. 70 is a root of the equation $f'(x) = 0$.

The figure in the preceding Article illustrates this theorem. We observe that between the two points of section A and B there are *three* maximum or minimum values, and between the two points B and C there is one such value. It appears also from the figure that the number of such values between two consecutive points of section of the axis is always odd.

Corollary. *Two consecutive roots of the derived equation may not comprise between them any root of the original equation, and never can comprise more than one.*

The first part of this proposition merely asserts that between two adjacent zero values of a polynomial there may be several maxima and minima; and the second part follows at once from the above theorem; for if two consecutive roots of $f'(x) = 0$ comprised between them more than one root of $f(x) = 0$, we should then have two consecutive roots of this latter equation comprising between them no root of $f'(x) = 0$, which is contradictory to the theorem.

72. Constitution of the Derived Functions. Let the roots of the equation $f(x) = 0$ be $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$. We have

$$f(x) \equiv (x - \alpha_1)(x - \alpha_2)(x - \alpha_3) \ldots (x - \alpha_n).$$

In this identical equation substitute $y+x$ for x;

$$f(y+x)=(y+x-\alpha_1)(y+x-\alpha_2)\ldots(y+x-\alpha_n)$$
$$=y^n+q_1y^{n-1}+q_2y^{n-2}+\ldots+q_{n-1}y+q_n,$$

where

$$q_1 \;=x-\alpha_1+x-\alpha_2+x-\alpha_3+\ldots+x-\alpha_n,$$
$$q_2 \;=(x-\alpha_1)(x-\alpha_2)+(x-\alpha_1)(x-\alpha_3)+\ldots+(x-\alpha_{n-1})(x-\alpha_n),$$

$$q_{n-1}=(x-\alpha_2)(x-\alpha_3)\ldots(x-\alpha_n)+(x-\alpha_1)(x-\alpha_3)\ldots(x-\alpha_n)+\ldots$$
$$+(x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_{n-1}),$$
$$q_n \;=(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)\ldots(x-\alpha_n).$$

We have, again,

$$f(y+x)=f(x)+f'(x)y+\frac{f''(x)}{1.2}y^2+\ldots+y^n.$$

Equating the two expressions for $f(y+x)$, we obtain

$f(x)=(x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_n)$,

$f'(x)=(x-\alpha_2)(x-\alpha_3)\ldots(x-\alpha_n)+\ldots\ldots$, as above written,

$\frac{f''(x)}{1.2}=$ the similar value of q_{n-2} in terms of x and the roots,

The value of $f'(x)$ may be conveniently written as follows :-

$$f'(x)=\frac{f(x)}{x-\alpha_1}+\frac{f(x)}{x-\alpha_2}+\ldots+\frac{f(x)}{x-\alpha_n}$$

73. Multiple Roots. Theorem. *A multiple root of the order m of the equation $f(x)=0$ is a multiple root of the order $m-1$ of the first derived equation $f'(x)=0$.*

This follows immediately from the expression given for $f'(x)$ in the preceding Article ; for if the factor $(x-\alpha_1)^m$ occurs in $f(x)$, *i.e.*, if $\alpha_1=\alpha_2=\ldots\ldots=\alpha_m$; we have

$$f'(x)=\frac{mf(x)}{x-\alpha_1}+\frac{f(x)}{x-\alpha_{m+1}}+\ldots+\frac{f(x)}{x-\alpha_n}.$$

Each term in this will still have $(x-\alpha_1)^m$ as a factor, except the first, which will have $(x-\alpha_1)^{m-1}$ as a factor ; hence $(x-\alpha_1)^{m-1}$ is a factor in $f'(x)$.

Corollary 1. *Any root which occurs m times in the equation $f(x)=0$ occurs in degrees of multiplicity diminishing by unity in the first $m-1$ derived equations.*

Since $f''(x)$ is derived from $f'(x)$ in the same manner as $f'(x)$ is from $f(x)$, it is evident by the theorem just proved that $f''(x)$ will contain $(x-\alpha_1)^{m-2}$ as a factor. The next derived function, $f'''(x)$, will contain $(x-\alpha_1)^{m-3}$; and so on.

Corollary 2. *If $f(x)$ and its first $m-1$ derived functions all vanish for a value α of x, then $(x-\alpha)^m$ is a factor in $f(x)$.*

This, which is the converse of the preceding corollary, is most readily established directly as follows :—Representing the derived functions by $f_1(x)$, $f_2(x)$,..., $f_{m-1}(x)$ (*see* Art. 6), and substituting $\alpha+x-\alpha$ for x, we find that $f(x)$ may be expanded in the form

$$f(\alpha)+f_1(\alpha)(x-\alpha)+\frac{f_2(\alpha)}{1\,.\,2}(x-\alpha)^2+\ldots+\frac{f_{m-1}(\alpha)}{1\,.\,2\ldots m-1}(x-\alpha)^{m-1}$$
$$+\frac{f_m(\alpha)}{1\,.\,2\ldots m}(x-\alpha)^m+\cdots+\frac{f_n(\alpha)}{1\,.\,2\ldots n}(x-\alpha)^n,$$

from which the proposition is manifest.

74. Determination of Multiple Roots. It is easily inferred from the preceding Article that if $f(x)$ and $f'(x)$ have a common factor $(x-\alpha)^{m-1}$, $(x-\alpha)^m$ will be a factor in $f(x)$; for, by Cor. 1, the $m-2$ next succeeding derived functions vanish as well as $f(x)$ and $f'(x)$ when $x=\alpha$; hence, by Cor. 2, α is a root of $f(x)$ of multiplicity m. In the same way it appears that if $f(x)$ and $f'(x)$ have other common factors

$$(x-\beta)^{p-1},\ (x-\gamma)^{q-1},\ (x-\delta)^{r-1},\ \text{etc.},$$

the equation $f(x)=0$ will have p roots equal to β, q roots equal to γ, r roots equal to δ, etc.

In order, therefore, to find whether any proposed equation has equal roots, and to determine such roots when they exist, we must find the greatest common measure of $f(x)$ and $f'(x)$. Let this be $\varphi(x)$. The determination of the equal roots will depend on the solution of the equation $\varphi(x)=0$.

Examples

1. Find the multiple roots of the equation

$$x^3+x^2-16x+20=0.$$

The G.C.M. of $f(x)$ and $f'(x)$ is easily found to be $x-2$; hence $(x-2)^2$ is a factor in $f(x)$. The other factor is $x+5$.

Whenever, after determining the multiple factors of $f(x)$, we wish to obtain the remaining factors, it will be found convenient to apply by repeated operations

the method of division of Art. 8. Here, for example, we divide twice by $x-2$ he calculation being represented as follows :—

$$\begin{array}{cccc}
1 & 1 & -16 & 20 \\
 & 2 & 6 & -20 \\
\hline
1 & 3 & -10 & 0 \\
 & 2 & 10 & \\
\hline
1 & 5 & 0 &
\end{array}$$

Thus 1 and 5 being the two co-efficients left, the third factor is $x+5$. This operation verifies the previous result, the remainders after each division vanishing as they ought.

2. Find the multiple roots, and the remaining factor, of the equation

$$x^5-10x^2+15x-6=0.$$

The G.C.M. of $f(x)$ and $f'(x)$ is found to be x^2-2x+1. Hence $(x-1)^3$ is factor in $f(x)$. Dividing three times in succession by $x-1$, we obtain

$$f(x)\equiv(x-1)^3(x^2+3x+6).$$

3. Find the multiple roots of the equation

$$x^4-2x^3-11x^2+12x+36=0.$$

The G.C.M. of $f(x)$ and $f'(x)$ is x^2-x-6. The factors of this are $x+2$ and $x-3$. Hence

$$f(x)\equiv(x+2)^2(x-3)^2.$$

4. Find all the factors of the polynomial

$$f(x)\equiv x^6-5x^5+5x^4+9x^3-14x^2-4x+8.$$

[*Ans.* $f(x)\equiv(x-1)(x+1)^2(x-2)^3$.

The ordinary process of finding the greatest common measure of a polynomial and its first derived function may become very laborious as the degree of the function increases. It is wrong, therefore, to speak, as is customary in works on the Theory of Equations, of the determination in this way of the multiple roots of numerical equations as a simple process, and one preliminary to further investigations relative to the roots. It is chiefly in connexion with Sturm's theorem that the operation is of any practical value. The further consideration of multiple roots is referred to Chap. X, where this theorem will be dicussed. It will be shown also in Chap. XI that the multiple roots of equations of degrees inferior to the sixth can, in any particular instance, be determined from simple considerations not involving the process of finding the greatest common measure.

75. This and the succeeding Article will be occupied with theorems which will be found of great importance in the subsequent discussion of methods of separating the roots of equations.

Theorem. *In passing continuously from a value $a-h$ of x a little less than a real root a of the equation $f(x)=0$ to a value $a+h$*

a little greater, the polynomials $f(x)$ and $f'(x)$ have unlike signs immediately before the passage through the root, and like signs immediately after.

Substituting $\alpha-h$ in $f(x)$ and $f'(x)$, and expanding, we have

$$f(\alpha-h)=f(\alpha)-f'(\alpha)h+\frac{f''(\alpha)}{1.2}h^2-\dots$$
$$f'(\alpha-h)=\qquad f'(\alpha)-f''(\alpha)h+\dots$$

Now, since $f(\alpha)=0$, the signs of these expressions, depending on those of their first terms, are unlike. When the sign of h is changed, the signs of the expressions become the same. The theorem is, therefore, proved.

Corollary. *The theorem remains true when α is a multiple root of any order of the equation $f(x)=0$.*

Let the root be repeated r times. The following functions (using suffixes in place of accents) all vanish :—

$$f(\alpha), f_1(\alpha), f_2(\alpha), \dots\dots, f_{r-1}(\alpha).$$

In the series for $f(\alpha-h)$ and $f'(\alpha-h)$ the first terms which do not vanish are, respectively,

$$\frac{f_r(\alpha)}{1\,.\,2\dots r}(-h)^r,\ \frac{f_r(\alpha)}{1\,.\,2\dots r-1}(-h)^{r-1}.$$

These clearly have unlike signs ; but when the sign of h is changed the signs of the terms will become the same. Hence the proposition is established.

76. Extending the reasoning of the last Article to every consecutive pair of the series

$$f(x), f_1(x), f_2(x), \dots\dots, f_{r-1}(x),$$

we may state the proposition generally as follows :—

Theorem. *When any equation $f(x)=0$ has an r-multiple root α, a value a little inferior to α gives to this series of r functions signs alternately positive and negative, or negative and positive ; and a value a little superior to it gives to all these functions the same sign ; and this sign is, moreover, the same as the sign of $f_r(\alpha)$, the first derived function which does not vanish when α is substituted for x.*

In order to give a precise idea of the use of this theorem, let us suppose that $f_5(\alpha)$ is the first function which does not vanish when α is substituted, and let its sign be negative ; the conclusion which may be drawn from the theorem is, that for a value $\alpha-h$ of x the signs of the series of functions $f, f_1, f_2, f_3, f_4, f_5$, are

$$+-+-+-\ ;$$

and for a value $\alpha+h$ of x they are

$$- - - - - - ;$$

for before the passage through the root the sign of f_4 must be different from that of f_5 ; the sign of f_3 must be different from that of f_4, and so on ; and after the passage the signs of all the functions must be the same. It is of course assumed here that h is so small that no root of $f_5(x)=0$ is included within the interval through which x travels.

Examples

1. Find the multiple roots of the equation

$$f(x) \equiv x^4+12x^3+32x^2-24x+4=0.$$

[*Ans.* $f(x) \equiv (x^2+6x-2)^2$.

2. Show that the binomial equation

$$x^n-a^n=0$$

cannot have equal roots.

3. Show that the equation

$$x^n-nqx+(n-1)r=0$$

will have a pair of equal roots if $q^n=r^{n-1}$.

4. Prove that the equation

$$x^5+5px^3+5p^2x+q=0$$

has a pair of equal roots when $q^2+4p^5=0$; and that if it has one pair of equal roots it must have a second pair.

5. Apply the method of Art. 74 to determine the condition that the cubic

$$z^3+3Hz+G=0$$

should have a pair of equal roots.

The last remainder in the process of finding the greatest common measure must vanish. [*Ans.* $G^2+4H^3=0$.

6. Apply the same method to show that both G and H vanish when the cubic has three equal roots.

7. If α, β, γ, δ be the roots of the biquadratic $f(x)=0$, prove that

$$f'(\alpha)+f'(\beta)+f'(\gamma)+f'(\delta)$$

can be expressed as a product of three factors.

[*Ans.* $(\alpha+\beta-\gamma-\delta)(\alpha+\gamma-\beta-\delta)(\alpha+\delta-\beta-\gamma)$.

8. If α, β, γ, δ, etc., be the roots of $f(x)=0$, and α', β', γ', etc., of $f'(x)=0$, prove

$$f'(\alpha)f'(\beta)f'(\gamma)f'(\delta)\ldots\ldots=n^n f(\alpha')f(\beta')f(\gamma')\ldots\ldots$$

and that each is equal to the absolute term in the equation whose roots are the squares of the differences.

9. If the equation

$$x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots\ldots+p_{n-1}x+p_n=0$$

have a double root α, prove that α is a root of the equation

$$p_1x^{n-1}+2p_2x^{n-2}+3p_3x^{n-3}+\ldots\ldots+np_n=0.$$

10. **Show that the max. and min. values of the cubic**

$$ax^3+3bx^2+3cx+d$$

are the roots of the equation

$$a^2\rho^2-2G\rho+\Delta=0,$$

where Δ is the discriminant.

If the curve representing the polynomial $f(x)$ be moved parallel to the axis of y (*see* Art. 10) through a distance equal to a max. or min. value ρ, the axis of x will become a tangent to it, *i.e.*, the equation $f(x)-\rho=0$ will have equal roots. Hence the max. and min. values are obtained by forming the discriminant of $f(x)-\rho$, or by putting $d-\rho$ for d in $G^2+4H^3=0$.

11. **Prove similarly that the max. and min. values of**

$$ax^4+4bx^3+6cx^2+4dx+e$$

are the roots of the equation

$$a^3\rho^3-3(a^2I-9H^2)\,\rho^2+3\,(aI^2-18HJ)\rho-\Delta=0,$$

where Δ is the discriminant of the quartic.

12. **Apply the theorem of Art. 76 to the function**

$$f(x)\equiv x^4-7x^3+15x^2-13x+4.$$

We have

$$f_1(x)=4x^3-21x^2+30x-13,$$
$$f_2(x)=2(6x^2-21x+15),$$
$$f_3(x)=2(12x-21),$$
$$f_4(x)=24.$$

Here $f_3(x)$ is the first function which does not vanish when $x=1$; and $f_3(1)$ is negative. What the theorem proves is, that for a value a little less than 1 the signs of f, f_1, f_2, f_3 are $+-+-$, and for a value a little greater than 1 they are all negative. We are able from this series of signs to trace the functions f, f_1, etc., in the neighbourhood of the point $x=1$. Thus the curve representing $f(x)$ is above the axis before reaching the multiple point $x=1$, and is below the axis immediately after reaching the point, and the axis must be regarded as cutting the curve in three coincident points, since $(x-1)^3$ is a factor in $f(x)$. Again, the curve corresponding to $f_1(x)$ is below the axis both before and after the passage through the point $x=1$. It touches the axis at that point. The curve representing $f_2(x)$ is above the axis before, and below the axis after, the passage, and cuts the axis at the point.

CHAPTER VIII

SYMMETRIC FUNCTIONS OF THE ROOTS

77. Newton's Theorem on the Sums of the Powers of the Roots. We now resume the discussion of symmetric functions of the roots of an equation, of which a short account has been previously given (*see* Art. 27); and proceed to prove certain general propositions relating to these functions :—

Prop. I. *The sums of the similar powers of the roots of an equation can be expressed rationally in terms of the coefficients.*

Let the equation be

$$\begin{aligned} f(x) &\equiv x^n + p_1 x^{n-1} + p_2 x^{n-2} + \dots\dots + p_n \\ &\equiv (x-\alpha_1)(x-\alpha_2)(x-\alpha_3)\dots\dots(x-\alpha_n) = 0. \end{aligned}$$

We proceed to calculate $\Sigma\alpha^2$, $\Sigma\alpha^3$,......, $\Sigma\alpha^m$; are, adopting the usual notation, $s_2, s_3, \dots\dots, s_m$, in terms of the coefficients $p_1, p_2, \dots$ ..., p_n.

We have, by Art. 72,

$$\begin{aligned} f'(x) &= \frac{f(x)}{x-\alpha_1} + \frac{f(x)}{x-\alpha_2} + \dots\dots + \frac{f(x)}{x-\alpha_n} \\ &\equiv nx^{n-1} + (n-1)p_1 x^{n-2} + (n-2)p_2 x^{n-3} + \dots + 2p_{n-2}x \\ &\qquad + p_{n-1}\,; \end{aligned}$$

and we find, dividing by the method of Art. 8,

$$\begin{array}{lll|l|l} \dfrac{f(x)}{x-\alpha} = x^{n-1} + \alpha & {}^{n-2} + \alpha^2 & & x^{n-3} + \alpha^3 & p^{n-4} + \dots + \alpha^{n-1} \\ \qquad\qquad\quad + p_1 & + p_1\alpha & & + p_1\alpha^2 & + p_1\alpha^{n-2} \\ & + p_2 & & + p_2\alpha & + p_2\alpha^{n-3} \\ & & & + p_3 & + \dots\dots \\ & & & & + p_{n-2}\alpha \\ & & & & + p_{n-1}. \end{array}$$

If, in this equation, we replace α by each of the quantities $\alpha_1, \alpha_2, \dots\dots, \alpha_n$ in succession, and put $s_p = \Sigma\alpha^p = \alpha_1{}^p + \alpha_2{}^p + \dots + \alpha_n{}^p$, we have, by adding all these results, the following value for $f'(x)$:—

$$\begin{array}{llll} f'(x) = nx^{n-1} + s_1 & x^{n-2} + s_2 & {}^{3} + s_3 & x^{n-4} + \dots\dots + s_{n-1} \\ \qquad\qquad\quad + np_1 & + p_1 s_1 & + p_1 s_2 & + p_1 s_{n-2} \\ & + np_2 & + p_2 s_1 & + p_2 s_{n-3} \\ & & + np_3 & \dots\dots \\ & & & + p_{n-2} s_1 \\ & & & + np_{n-1}\,; \end{array}$$

whence, comparing this value of $f'(x)$ with the former, we obtain the following relations :—

$$s_1+p_1=0,$$
$$s_2+p_1s_1+2p_2=0,$$
$$s_3+p_1s_2+p_2s_1+3p_3=0,$$
$$s_4+p_1s_3+p_2s_2+p_3s_1+4p_4=0,$$

$$s_{n-1}+p_1s_{n-2}+p_2s_{n-3}+\cdots+p_{n-2}s_1+(n-1)p_{n-1}=0.$$

The first equation determines s_1 in terms of $p_1, p_2, \ldots\ldots p_n$; the second s_2 ; the third s_3 ; and so on, until s_{n-1} is determined. We find in this way

$$s_1=-p_1,\ s_2=p_1^2-2p_2,\ s_3=-p_1^3+3p_1p_2-3p_3,$$
$$s_4=\ p_1^4-4p_1^2p_2+4p_1p_3+2p_2^2-4p_4,$$
$$s_5=-p_1^5+5p_1^3p_2-5p_1^2p_3-5(p_2^2-p_4)p_1+5(p_2p_3-p_5)\text{ ; etc.}$$

Having shown how $s_1, s_2, s_3, \ldots\ldots, s_{n-1}$ can be calculated in terms of the coefficients, we proceed now to extend our results to the sums of all positive powers of the roots, *viz.*, $s_n, s_{n+1}, \ldots\ldots, s_m$. For this purpose we have

$$x^{m-n}f(x)\equiv x^m+p_1x^{m-1}+p_2x^{m-2}+\ldots\ldots+p_nx^{m-n}.$$

Replacing, in this identity, x by the roots $\alpha_1, \alpha_2, \ldots\ldots, \alpha_n$, in succession, and adding, we have

$$s_m+p_1s_{m-1}+p_2s_{m-2}+\ldots\ldots+p_ns_{m-n}=0.$$

Now, giving m the values n, $n+1$, $n+2$, etc., successively, and observing that $s_0=n$, we obtain from the last equation

$$s_n\ \ +p_1s_{n-1}+p_2s_{n-2}+\ldots\ldots+np_n=0,$$
$$s_{n+1}+p_1s_n\ \ +p_2s_{n-1}+\ldots\ldots+p_ns_1=0,$$
$$s_{n+2}+p_1s_{n+1}+p_2s_n\ \ +\ldots\ldots+p_ns_2=0,\text{ etc.}$$

Hence the sums of all positive powers of the roots may be expressed by integral functions of the coefficients. And by transforming the equation into one whose roots are the reciprocals of $\alpha_1, \alpha_2, \alpha_3, \ldots\ldots, \alpha_n$, and applying the above formulas, we may express similarly all negative powers of the roots.

78. Prop. II. *Every rational symmetric function of the roots of an algebraic equation can be expressed rationally in terms of the coefficients.*

It is sufficient to prove this theorem for integral functions only since fractional symmetric functions can be reduced to a single fraction whose numerator and denominator are both integral symmetric functions. Every integral function of $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ is the sum of a number of terms of the form $N\alpha_1^p\alpha_2^q\alpha_3^r\ldots$, where N is a numerical

constant ; and if this function be symmetrical we can write it under the form $N\Sigma\alpha_1{}^p\alpha_2{}^q\alpha_3{}^r\ldots$, all the terms being of the same type. Therefore, if we prove that this quantity can be expressed rationally in terms of the coefficients, the theorem will be demonstrated. We shall first establish the following value of the symmetric function $\Sigma\alpha_1{}^p\alpha_2{}^q$:—

$$\Sigma\alpha_1{}^p\alpha_2{}^q = s_p s_q - s_{p+q}. \qquad \ldots(1)$$

To prove this, we multiply together s_p and s_q, where

$$s_p = \alpha_1{}^p + \alpha_2{}^p + \alpha_3{}^p + \ldots \alpha_n{}^p,$$
$$s_q = \alpha_1{}^q + \alpha_2{}^q + \alpha_3{}^q + \ldots \alpha_n{}^q,$$

whence

$$s_p s_q = \alpha_1{}^{p+q} + \alpha_2{}^{p+q} + \ldots + \alpha_n{}^{p+q} + \alpha_1{}^p\alpha_2{}^q + \alpha_1{}^q\alpha_2{}^p + \text{etc.},$$

or $\qquad s_p s_q = s_{p+q} + \Sigma\alpha_1{}^p\alpha_2{}^q,$

which expresses the double function $\Sigma\alpha_1{}^p\alpha_2{}^q$ in terms of the single functions s_p, s_q, s_{p+q}, in the form above written.

We proceed now to prove a similar expression for the triple function, *viz.*,

$$\Sigma\alpha_1{}^p\alpha_2{}^q\alpha_3{}^r = s_p s_q s_r - s_{q+r}s_p - s_{r+p}s_q - s_{p+q}s_r + 2s_{p+q+r} \qquad \ldots(2)$$

Multiplying together $\Sigma\alpha_1{}^p\alpha_2{}^q$ and s_r, where

$$\Sigma\alpha_1{}^p\alpha_2{}^q = \alpha_1{}^p\alpha_2{}^q + \alpha_1{}^q\alpha_2{}^p + \alpha_1{}^p\alpha_3{}^q + \ldots$$
$$s_r = \alpha_1{}^r + \alpha_2{}^r + \alpha_3{}^r + \ldots + \alpha_n{}^r,$$

we obtain an expression consisting of three different parts, *viz.*, terms of the form $\quad \Sigma\alpha_1{}^{p+r}\alpha_2{}^q$, $\Sigma\alpha_1{}^{q+r}\alpha_2{}^p$, and $\Sigma\alpha_1{}^p\alpha_2{}^q\alpha_3{}^r$.

Hence

$$s_r\Sigma\alpha_1{}^p\alpha_2{}^q = \Sigma\alpha_1{}^{p+r}\alpha_2{}^q + \Sigma\alpha_1{}^{q+r}\alpha_2{}^p + \Sigma\alpha_1{}^p\alpha_2{}^q\alpha_3{}^r,$$

a formula connecting double and triple symmetric functions.

But, by (1),

$$\Sigma\alpha_1{}^{p+r}\alpha_2{}^q = s_{p+r}s_q - s_{p+q+r},$$
$$\Sigma\alpha_1{}^{q+r}\alpha_2{}^p = s_{q+r}s^p - s_{p+q+r},$$
$$\Sigma\alpha_1{}^p\alpha_2{}^q \ = s_p s_q \ \ - s_{p+q}.$$

Substituting these values, we find the triple function $\Sigma\alpha_1{}^p\alpha_2{}^q\alpha_3{}^r$ expressed as above in terms of single functions in the series s_1, s_2, s_3, etc.

In the same manner the quadruple function $\Sigma\alpha_1{}^p\alpha_2{}^q\alpha_3{}^r\alpha_4{}^s$. can be made to depend on the triple function $\Sigma\alpha_1{}^p\alpha_2{}^q\alpha_3{}^r$, and ultimately on s_1, s_2, s_3, etc. ; and so on. Whence, finally, every rational symmetric function of the roots may be expressed in terms of the coefficients, since, by Prop. I., s_1, s_2, s_3, etc., can be so expressed.

The formulas (1) and (2) require to be modified when any of the exponents become equal.

Thus, if $p=q$, $\alpha_1^p\alpha_2^q \equiv \alpha_2^p\alpha_1^q$, and the terms in (1) become equal two and two ; therefore, $\Sigma\alpha_1^p\alpha_2^q = 2\Sigma\alpha_1^p\alpha_2^p$; whence

$$\Sigma\alpha_1^p\alpha_2^p = \tfrac{1}{2}(s_p^2 - s_{2p}).$$

Similarly, if $p=q=r$ in $\Sigma\alpha_1^p\alpha_2^q\alpha_3^r$, the six terms obtained by interchanging the roots in $\alpha_1^p\alpha_2^q\alpha_3^r$ become all equal ; hence

$$\Sigma\alpha_1^p\alpha_2^p\alpha_3^p = \frac{1}{2.3}(s_p^3 - 3s_p s_{2p} + 2s_{3p}).$$

And, in general, if exponents become equal, each term is repeated 1 . 2 . 3... t times.

Examples

1. Prove

$$\Sigma\alpha_1^p\alpha_2^q\alpha_3^r\alpha_4^s = s_p s_q s_r s_s - \Sigma s_p s_q s_{r+s} + 2\Sigma s_p s_{q+r+s} + \Sigma s_{p+q} s_{r+s} - 6s_{p+q+r+s}$$

2. Prove

$$24\Sigma\alpha_1^m\alpha_2^m\alpha_3^m\alpha_4^m = s_m^4 - 6s_m^2 s_{2m} + 8s_m s_{3m} + 3s_{2m}^2 - 6s_{4m}.$$

79. Prop. III. *The value of s_r, expressed in terms of p_1 p_2, ...p_n, is the coefficient of y^r in the expansion, by ascending powers of y, of* $-r \log y^n \left(\frac{1}{y}\right)$.

Since

$$x^n + p_1{}^{n-1} + p_2 x^{n-2} + \ldots + p_n \equiv (x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_n),$$

putting $\frac{1}{y}$ for x in this identical equation, we find

$$1 + p_1 y + p_2 y^2 + p_3 y^3 + \ldots + p_n y^n \equiv (1-\alpha_1 y)(1-\alpha_2 y)\ldots(1-\alpha_n y).$$

Now, taking the Napierian logarithms of both sides,

$$\begin{array}{l|l|l|l}
p_1 y + p_2 & y^2 + p_3 & y^3 + y_4 & y^4 + p_5 \quad y^5 + \text{etc.}, \ldots + P_r y^r + \text{etc}. \\
-\frac{1}{2}p_1^2 & -p_1p_2 & -p_1p_3 & -p_2p_3 \\
 & +\frac{1}{3}p_1^3 & +\frac{1}{2}p_2^2 & +p_1p_2^2 \\
 & & +p_1^2p_2 & +p_1^2p_3 \\
 & & -\frac{1}{4}p_1^4 & -p_1p_4 \\
 & & & -p_1^3p_2 \\
 & & & +\frac{1}{5}p_1^5
\end{array}$$

$$= -ys_1 - \frac{1}{2}y^2s_2 - \frac{1}{3}y^3s_3 - \ldots - \frac{1}{r}y^rs_r - \text{etc.}$$

Therefore, equating coefficients of y^r in both expansions,

$$s_r = -r P_r,$$

where P_r is the coefficient of y^r in $\log y^n f\left(\frac{1}{y}\right)$.

From the above identical equation it may be seen that s_r (r less than n) involves the coefficients $p_1, p_2, p_3, \ldots, p_r$ only; and, therefore, $p_{r+1}, p_{r+2}, \ldots, p_n$ may be made to vanish without affecting the form of the expression of s_r in terms of the coefficients.

80. *To express the coefficients in terms of the sums of the powers of the roots.*

Since

$$1 + p_1 y + p_2 y^2 + \ldots + p_n y^n \equiv (1 - \alpha_1 y)(1 - \alpha_2 y) \ldots (1 - \alpha_n y),$$

we have

$$\log\,(1 + p_1 y + \ldots + p_n y^n) \equiv -y s_1 - \frac{1}{2}\,y^2 s_2 - \ldots - \frac{1}{r}\,y^r s_r - \ldots; \qquad \ldots(1)$$

and, therefore,

$$1 + p_1 y + p_2 y^2 + \ldots + p_n y^n \equiv e^{-y s_1 - \frac{1}{2} y^2 s_2 - \frac{1}{3} y^3 s_3 - \ldots},$$

which becomes by expansion

$$\begin{array}{llll}
1 - s_1 y - \frac{1}{2} s_2 & y^2 - \frac{1}{2} s_3 & y^3 - \frac{1}{4} s_4 & y^4 - \ldots \\
\quad + \frac{1}{1.2} s_1 & \quad + \frac{1}{1.2} s_1 s_2 & \quad + \frac{1}{3} s_1 s_3 & \\
 & \quad \frac{1}{1.2.3} s_1^3 & \quad - \frac{1}{4} s_1^2 s_2 & \\
 & & \quad + \frac{1}{2.4} & \\
 & & \quad + \frac{1}{2.3.4} s_1^4 &
\end{array}$$

Now, comparing the coefficients of the different powers of y, we obtain values for $p_1, p_2, p_3, \ldots, p_n$, in terms of $s_1, s_2, \ldots s_n$; and we see that p_r involves no sum of powers beyond s_r.

If the identity (1) be differentiated with regard to y, the equations of Art. 77 connecting the coefficients and sums of powers may be derived immediately from the resulting identity.

It is important to observe that the problem to express any symmetric function to the roots in terms of the coefficients, or any

coefficient in terms of the sums of the powers of the roots, is perfectly definite, there being only one solution in each case.

General expressions, due to Waring, for s_m in terms of the coefficients, and for p_m in terms of the sums of the powers of the roots, will be given in a subsequent chapter.

Examples

1. Determine the value of

$$\varphi(\alpha_1)+\varphi(\alpha_2)+\dots+\varphi(\alpha_n),$$

where $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n$ are the roots of $f(x)=0$, and $\varphi(x)$ is any rational and integral function of x.

We have

$$\frac{f'(x)}{f(x)}=\frac{1}{x-\alpha_1}+\frac{1}{x-\alpha_2}+\dots+\frac{1}{x-\alpha_n},$$

and

$$\frac{f'(x)\varphi(x)}{f(x)}=\frac{\varphi(x)}{x-\alpha_1}+\frac{\varphi(x)}{x-\alpha_2}+\dots+\frac{\varphi(x)}{x-x_n}.$$

Performing the division, and retaining only the remainders on both sides of this equation, we have

$$\frac{R_0x^{n-1}+R_1x^{n-2}+\dots+R_{n-1}}{f(x)}=\frac{\varphi(\alpha_1)}{x-\alpha_1}+\frac{\varphi(\alpha_2)}{x-\alpha_2}+\dots+\frac{\varphi(\alpha_n)}{x-\alpha_n};$$

whence $R_0x^{n-1}+R_1x^{n-2}+\dots+R_{n-1}=\Sigma\varphi(\alpha_1)(x-\alpha_2)(x-\alpha_3)\dots(x-\alpha_n)$;

and, comparing the coefficients of x^{n-1} on both sides of the equation,

$$R_0=\Sigma\varphi(\alpha_1).$$

2. Prove that s_p is the coefficient of $\frac{1}{x^{p+1}}$ in the quotient of the division of $f'(x)$ by $f(x)$ arranged according to negative powers of x.

3. Prove that s_{-p} is the coefficient (with sign changed) of x^{p-1} in the same quotient arranged according to positive powers of x.

4. If the degree of $\varphi(x)$ does not exceed $n-2$, prove

$$\sum_{r=1}^{r=n}\frac{\varphi(\alpha_r)}{f'(\alpha_r)}=0,$$

where $\sum_{r=1}^{r=n}$ denotes the sum obtained by giving r all values from 1 to n inclusive.

We have, by partial fractions,

$$\frac{\varphi(x)}{f(x)}=\frac{A_1}{x-\alpha_1}+\frac{A_2}{x-\alpha_2}+\dots+\frac{\dots}{x-\alpha_n};$$

and, multiplying across by $f(x)$, and putting $x=\alpha_1, \alpha_2, \dots\dots$ in succession,

$$\frac{\varphi(x)}{x}=\frac{\varphi(\alpha_1)}{f'(\alpha_1)}\frac{1}{x-\alpha_1}+\frac{\varphi(\alpha_2)}{f'(\alpha_2)}\frac{1}{x-\alpha_2}+\dots+\frac{\varphi(\alpha_n)}{f'(\alpha_n)}\frac{1}{x-\alpha_n};$$

whence

$$\frac{x\varphi(x)}{f(x)} = \sum_{r=1}^{r=n} \frac{\varphi(\alpha_r)}{f'(\alpha_r)}\left(1+\frac{\alpha_r}{x}+\frac{\alpha_r^2}{x^2}+\ldots\right).$$

When $\varphi(x)$ is of the degree $n-2$; expressing the first side of the equation as a function of $\frac{1}{x}$, it readily appears that there is no term without $\frac{1}{x}$ as a multiplier. We have, therefore, comparing coefficients,

$$\sum_{r=1}^{r=n} \frac{\varphi(\alpha_r)}{f'(\alpha_r)}=0.$$

As φ may be any rational and integral function of degree not higher than $n-2$, we have the following particular cases which are worthy of special notice :—

$$\Sigma\frac{\alpha^{n-2}}{f'(\alpha)}=0,\ \Sigma\frac{\alpha^{n-3}}{f'(\alpha)}=0\ \ldots\ \frac{\alpha}{f'(\alpha)}=0,\ \Sigma\frac{1}{f'(\alpha)}=0.$$

9. Given the following $n-2$ equations between n variables $x_1, x_2, \ldots x_n$:—

$$\sum_{r=1}^{r=n} x_r=0,\ \sum_{r=1}^{r=n} \alpha_r x_r=0, \ldots \sum_{r=1}^{r=n} \alpha_r^{n-3}x_r=0,$$

express the n variables in terms of two new variable X_1, X_2.

[*Ans.* $x_r=\dfrac{X_1+\alpha_r X_2}{f'(\alpha_r)}$.

81. Order and Weight of Symmetric Functions. The degree in *all* the roots of any term of a symmetric function of the roots [*see* Art. 21] is called the *weight* of the function. The highest degree in which each root enters the function is called its *order*. The weight, for example, of $\Sigma\alpha\beta^2\gamma^3$ is six, and its order three. It has been proved (Art. 28) that in the value in the terms of the coefficients of any symmetric function of the roots the sum of the suffixes in each term is equal to the weight of the function. We now prove another proposition relating to symmetric functions, *viz.*,—*The degree in terms of the coefficients $p_1, p_2, \ldots, p_n$ of the value of any symmetric function is equal to the order of symmetric function.*

This can be readily inferred from the equation of Art. 23, since the value of each coefficient in terms of the roots contains any root in the first power only, and, therefore, the highest degree in the coefficients will be the same as the degree of the corresponding symmetric function in any individual root. The value of $\Sigma\alpha^2\beta^2$, for example, is $p_2^2-2p_1p_3+2p_4$. The degree of this function of the coefficients is two, which is also the order of the symmetric function.

As the proposition just stated is of importance, we add another proof, in which the symmetric function multiplied by a suitable power of a_0 is expressed as a homogeneous and integral function of the coefficients $a_0, a_1, a_2, \ldots, a_n$, the form in which the result will usually appear in subsequent applications.

Replace the coefficients

$$p_1, p_2, \ldots, p_n \text{ by } \frac{a_1}{a_0}, \frac{a_2}{a_0}, \ldots, \frac{a_n}{a_0}.$$

Now, if $\varphi(\alpha_1, \alpha_2, \ldots, \alpha_n)$ denote any rational and integral symmetric function of the roots, we have

$$a_0^h \varphi(\alpha_1, \alpha_2, \ldots, \alpha_n) = F(a_0, a_1, a_2, \ldots, a_n),$$

where h is the degree, in the coefficients, of $F(a_0, a_1, a_2, \ldots, a_n)$, a homogeneous and integral function of the coefficients not divisible by a_0.

We require now to show that h is the order of φ. For this purpose change the roots into their reciprocals, and, therefore, $a_0, a_1, \ldots, a_n$ into $a_n, a_{n-1}, \ldots\ldots, a_0$. Whence

$$a_n^h \varphi\left(\frac{1}{\alpha_1}, \frac{1}{\alpha_2}, \ldots, \frac{1}{\alpha_n}\right) = F(a_n, a_{n-1}, a_{n-2}, \ldots, a_0); \qquad \ldots(1)$$

also

$$\varphi\left(\frac{1}{\alpha_1}, \frac{1}{\alpha_2}, \ldots\ldots, \frac{1}{\alpha_n}\right) = \frac{\psi(\alpha_1, \alpha_2, \alpha_3, \ldots\ldots, \alpha_n)}{(\alpha_1\alpha_2\alpha_3\ldots\ldots\alpha_n)^p},$$

where p is the order of φ, and ψ an integral function not divisible by the product of all the roots; $(\alpha_1\alpha_2\alpha_3\ldots\alpha_n)^p$ being the lowest common denominator of all the terms. Substituting in (1), we have

$$a_0^p \psi(\alpha_1, \alpha_2, \ldots, \alpha_n) = \pm a_n^{p-h} F(a_n, a_{n-1}, \ldots, a_0).$$

From this equation it follows that p is equal to h; for if p were greater than h, $\psi(\alpha_1, \alpha_2, \ldots\ldots, \alpha_n)$ would be divisible by the product $\alpha_1\alpha_2\ldots\ldots\alpha_n$, and if it were less, the function of the coefficients $F(a_n, a_{n-1}, \ldots, a_0)$ would be divisible by a_n, both of which suppositions are contrary to hypothesis.

82. Calculation of Symmetric Functions of the Roots. Any rational symmetric function can be calculated by aid of the proposition of Art. 78. In practice, however, other methods are usually more convenient, as will appear from the examples which follow the present Article. We shall show also, when this subject is resumed in the second volume of this work, that use may be made of methods founded on the principles there explained to facilitate in many instances the calculation of symmetric functions.

The number of terms in any symmetric function of the roots is easily determined. For example, the number of terms in $\Sigma\alpha_1^3\alpha_2^2\alpha_3$ of the equation of the n^{th} degree in $n(n-1)(n-2)$, this being the number of permutations of n things taken three together. If the exponents of the roots in any term be not all different, the number of

terms will be reduced. Thus $\Sigma\alpha^2\beta\gamma$ for a biquadratic consists of twelve terms only (*see* Ex. 6, p. 39), and not of twenty-four, since the two permutations $\alpha\beta\gamma$, $\alpha\gamma\beta$ give only one distinct term, *viz.*, $\alpha^2\beta\gamma$, in $\Sigma\alpha^2\beta\gamma$. The student acquainted with the theory of permutations will have no difficulty in effecting these reductions in any particular case. When two exponents of roots are equal, the number obtained on the supposition that they are all unequal is to be divided by 1.2 ; when three become equal this number is to be divided by 1.2.3 ; and so on. In general, the number of terms in $\Sigma\alpha_1^{p}\alpha_2^{q}\alpha_3^{r},\ldots\ldots$ of the equation of the n^{th} degree, each term containing m roots, and ν of the indices being equal, is

$$\frac{n(n-1)(n-2)\ldots(n-m+1)}{1.2.3\ldots\nu}.$$

When the highest power in which any one root enters into the symmetric function of the roots is small, *i.e.*, when the order of the function (*see* Art. 81) is low, the methods already illustrated in Art. 27 may be employed with advantage for the calculation of the symmetric function.

It is important to observe that when any symmetric function, whose degree in all the roots (*i.e.*, its weight) is n, is calculated in terms of the co-efficients $p_1, p_2, \ldots, p_n$ for the equation of the n^{th} degree, its value for an equation of any higher degree (the numerical co-efficients being all equal to unity) is precisely the same ; for it is clear that no co-efficient beyond p_n can enter into this value, and the equation of Art. 77, by means of which the calculations can be supposed to be made, have precisely the same form for an equation of the n^{th} degree as for equations of all higher degrees. It is also evident that the value of the same symmetric function for an equation of a degree m (lower than n) is obtained by putting $p_{m+1}, p_{m+2}, \ldots, p_n$ all equal to zero in the calculated value for an equation of the n^{th} degree, since the equation of lower degree can be derived from that of the n^{th} by putting the co-efficients beyond p_m equal to zero ; and the corresponding symmetric function reduces similarly by putting the roots $\alpha_{m+1}, \alpha_{m+2}, \ldots\ldots, \alpha_n$ each equal to zero.

Examples

1. Calculate $\Sigma\alpha_1^2\alpha_2\alpha_3$ of the roots of the equation

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots\ldots + p_{n-1}x + p_n = 0.$$

Multiply together the equations

$$\Sigma\alpha_1 \quad = -p_1,$$
$$\Sigma\alpha_1\alpha_2\alpha_3 = -p_3.$$

In the product the term $\alpha_1^2\alpha_2\alpha_3$ occurs only once; the term $\alpha_1\alpha_2\alpha_3\alpha_4$ occurs four times, arising from the product of α_1 by $\alpha_2\alpha_3\alpha_4$, of α_2 by $\alpha_1\alpha_3\alpha_4$, of α_3 by $\alpha_1\alpha_2\alpha_4$ and of α_4 by $\alpha_1\alpha_2\alpha_3$. Hence

$$\Sigma\alpha_1^2\alpha_2\alpha_3+4\Sigma\alpha_1\alpha_2\alpha_3\alpha_4=p_1p_3;$$

therefore

$$\Sigma\alpha_1^2\alpha_2\alpha_3=p_1p_3-4p_4. \qquad \text{(Compare Ex. 6, Art. 27.)}$$

If the calculation were conducted by the method of Art. 78, we should have

$$\Sigma\alpha_1^2\alpha_2\alpha_3=\tfrac{1}{2}s_2s_1^2-s_1s_3-\tfrac{1}{2}s_2^2+s_4,$$

which leads, on substituting the values of Art. 77, to the same result; but it is clear that in this case the former process is much more simple, since the values of s_1, s_2, etc., introduce a number of terms which destroy one another.

2. Calculate $\Sigma\alpha_1^2\alpha_2^2$ for the general equation.

Squaring

$$\Sigma\alpha_1\alpha_2=p_2,$$

$$\Sigma\alpha_1^2\alpha_2^2+2\Sigma\alpha_1^2\alpha_2\alpha_3+6\Sigma\alpha_1\alpha_2\alpha_3\alpha_4=p_2^2.$$

In squaring it is evident that the term $\alpha_1\alpha_2\alpha_3\alpha_4$ will arise from the product of $\alpha_1\alpha_2$ by $\alpha_3\alpha_4$, or of $\alpha_1\alpha_3$ by $\alpha_2\alpha_4$, or of $\alpha_1\alpha_4$ by $\alpha_2\alpha_3$; hence the coefficient of $\alpha_1\alpha_2\alpha_3\alpha_4$ in the result is 6, since each of these occurs twice in the square. The result differs from the similar equation of Ex. 8, Art. 27, only in having Σ before the term $\alpha_1\alpha_2\alpha_3\alpha_4$. Hence, finally,

$$\Sigma\alpha_1^2\alpha_2^2=p_2^2-2p_1p_3+2p_4.$$

3. Calculate $\Sigma\alpha_1^3\alpha_2$ for the general equation.

We have, as in Ex. 9, Art. 27,

$$\Sigma\alpha_1^2\Sigma\alpha_1\alpha_2=\Sigma\alpha_1^3\alpha_2+\Sigma\alpha_1^2\alpha_2\alpha_3,$$

Hence, employing previous results,

$$\Sigma\alpha_1^3\alpha_2=p_1^2p_2-2p_2^2-p_1p_3+4p_4.$$

4. Calculate $\Sigma\alpha_1^2\alpha_2^2\alpha_3$ for the general equation.

The result will be the same as if the calculations were made for the equation of the fifth degree.

To obtain the symmetric function we multiply together $\Sigma\alpha_1\alpha_2$ and $\Sigma\alpha_1\alpha_2\alpha_3$; and consider what types of terms, involving the five roots α_1, α_2, α_3, α_4, α_5, can result. The term $\alpha_1^2\alpha_2^2\alpha_3$ will occur only once in the product, since it can only arise by multiplying $\alpha_1\alpha_2$ by $\alpha_1\alpha_2\alpha_3$. Terms of the type $\alpha_1^2\alpha_2\alpha_3\alpha_4$ will occur each three times; since $\alpha_1^2\alpha_2\alpha_3\alpha_4$ will arise from the product of $\alpha_1\alpha_2$ by $\alpha_1\alpha_3\alpha_4$, of $\alpha_1\alpha_3$ by $\alpha_1\alpha_2\alpha_4$, or of $\alpha_1\alpha_4$ by $\alpha_1\alpha_2\alpha_3$; and it cannot arise in any other way. The term $\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5$ will occur ten times in the product, since it will arise from the product of any pair by the other three roots, and there are ten combinations in pairs of the five roots. We have, then, for the general equation,

$$\Sigma\alpha_1\alpha_2\ \Sigma\alpha_1\alpha_2\alpha_3=\Sigma\alpha_1^2\alpha_2^2\alpha_3+3\Sigma\alpha_1^2\alpha_2\alpha_3\alpha_4+10\Sigma\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5.$$

[We can verify this equation when $n=5$, just as in Ex. 9, Art. 27; for the product of two factors, each consisting of 10 terms, will contain 100 terms. These are made up of the 30 terms contained in $\Sigma\alpha_1^2\alpha_2^2\alpha_3$ along with the 20 terms contained in $\Sigma\alpha_1^2\alpha_2\alpha_3\alpha_4$, each taken three times, and the term $\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5$ taken 10 times.]

Thus the calculation of the required symmetric function involves that of $\Sigma\alpha_1^2\alpha_2\alpha_3\alpha_4$; for which we easily find

$$\Sigma\alpha_1 \; \Sigma\alpha_1\alpha_2\alpha_3\alpha_4 = \Sigma\alpha_1^2\alpha_2\alpha_3\alpha_4 + 5\Sigma\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5.$$

Hence, finally, we obtain

$$\Sigma\alpha_1^2\alpha_2^2\alpha_3 = -p_2p_3 + 3p_1p_4 - 5p_5.$$

The process of Art. 78 would involve the calculation of s_5; and many terms would be introduced through the values of s_1, s_2, etc., which disappear in the result.

5. Find the value of $\Sigma\alpha_1^2\alpha_2^2\alpha_3\alpha_4$ for the general equation.

We multiply together $\Sigma\alpha_1\alpha_2$ and $\Sigma\alpha_1\alpha_2\alpha_3\alpha_4$, and consider what types of terms can arise involving the six roots $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6$. The term $\alpha_1^2\alpha_2^2\alpha_3\alpha_4$ can occur only once. Terms of the type $\alpha_1^2\alpha_2\alpha_3\alpha_4\alpha_5$ will each occur four times, this term arising from the product of $\alpha_1\alpha_2$ by $\alpha_1\alpha_3\alpha_4\alpha_5$ or of $\alpha_1\alpha_3$ by $\alpha_1\alpha_2\alpha_4\alpha_5$, or of $\alpha_1\alpha_4$ by $\alpha_1\alpha_2\alpha_3\alpha_5$, or of $\alpha_1\alpha_5$ by $\alpha_1\alpha_2\alpha_3\alpha_4$. The term $\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6$ will occur fifteen times, this being the number of combination in pairs of the six roots. Hence

$$\Sigma\alpha_1\alpha_2 \; \Sigma\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5 = \Sigma\alpha_1^2\alpha_2^2\alpha_3\alpha_4 + 4\Sigma\alpha_1^2\alpha_2\alpha_3\alpha_4\alpha_5 + 15\Sigma\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6$$

We have again, for the calculation of $\Sigma\alpha_1^2\alpha_2\alpha_3\alpha_4\alpha_5$,

$$\Sigma\alpha_1 \; \Sigma\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5 = \Sigma\alpha_1^2\alpha_2\alpha_3\alpha_4\alpha_5 + 6\Sigma\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6$$

Hence, finally,

$$\Sigma\alpha_1^2\alpha_2^2\alpha_3\alpha_4 = p_2p_4 - 4p_1p_5 + 9p_6.$$

6. Find the value of $\Sigma\alpha_1^2\alpha_2^2\alpha_3^2$ in terms of the co-efficients of the general equation.

Squaring $\Sigma\alpha_1\alpha_2\alpha_3$, we have

$$\Sigma\alpha_1\alpha_2\alpha_3 \; \Sigma\alpha_1\alpha_2\alpha_3 = \Sigma\alpha_1^2\alpha_2^2\alpha_3^2 + 2\Sigma\alpha_1^2\alpha_2^2\alpha_3\alpha_4 + 6\Sigma\alpha_1^2\alpha_2\alpha_3\alpha_4\alpha_5 + 20\Sigma\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6$$

from which we find

$$\Sigma\alpha_1^2\alpha_2^2\alpha_3^2 = p_3^2 - 2p_2p_4 + 2p_1p_5 - 2p_6.$$

83. Homogeneous Products. There are, in general, several symmetric functions of n quantities $\alpha_1, \alpha_2, \ldots, \alpha_n$ which have the same weight, and amongst these may be included two or more which have the same order as well as weight. Of any n letters there are, for example, the following symmetric functions whose weight is four :—

$$\Sigma\alpha_1^4,\ \Sigma\alpha_1^3\alpha_2,\ \Sigma\alpha_1^2\alpha_2^2,\ \Sigma\alpha_1^2\alpha_2\alpha_3,\ \Sigma\alpha_1\alpha_2\alpha_3\alpha_4.$$

The sum of all such symmetric functions of weight r is called the "sum of the homogeneous products of r dimensions" of the n letters. This sum we denote by Π_r. It is easy to see that Π_r is the co-efficient of x^r in the following product of n factors :—

$$(1+\alpha_1x+\alpha_1^2x^2+\ldots)(1+\alpha_2x+\alpha_2^2x^2+\ldots)\ldots(1+\alpha_nx+\alpha_n^2x^2+\ldots).$$

The examples which follow include the most fundamental propositions connecting the sums of the homogeneous products with the co-efficients of the equation whose roots are $\alpha_1, \alpha_2, \ldots, \alpha_n$.

Examples

1. Prove

$$\pi_r = \Sigma \frac{\alpha^{n+r-1}}{f'(\alpha)}.$$

We have

$$\frac{x^n}{f(x)} = \frac{1}{(1-\alpha_1 y)(1-\alpha_2 y)\dots(1-\alpha_n y)}, \text{ where } y = \frac{1}{x},$$

$$= (1+\alpha_1 y+\alpha_1^2 y^2+\dots)(1+\alpha_2 y+\alpha_2^2 y^2+\dots)\dots(1+\alpha_n y+\alpha_n^2 y^2+\dots)$$

$$\equiv 1+\pi_1 y+\pi_2 y^2+\dots\pi_r y^r+\dots$$

Also
$$\frac{x^{n-1}}{f(x)} = \Sigma \frac{\alpha^{n-1}}{f'(\alpha)} \cdot \frac{1}{x-\alpha},$$

whence
$$\frac{x^n}{f(x)} = \Sigma \frac{\alpha^{n-1}}{f'(\alpha)} \cdot \frac{1}{1-\alpha y} = \Sigma \frac{\alpha^{n+r-1}}{f'(\alpha)} y^r;$$

from which the result follows by comparing co-efficients of y^r.

2. Express the sums of the homogeneous product of the roots in terms of the co-efficients of an equation, and *vice versa*.

Since

$$(1-\alpha_1 y)(1-\alpha_2 y)\dots(1-\alpha_n y) = 1+p_1 y+p_2 y^2+\dots+p_n y^n,$$

we have immediately, from the preceding example,

$$(1+p_1 y+p_2 y^2+\dots+p_n y^n)(1+\pi_1 y+\pi_2 y^2+\dots) \equiv 1,$$

whence

$$p_1+\pi_1=0,\ p_2+\pi_2+p_1\pi_1=0,\ p_3+\pi_3+p_1\pi_2+p_2\pi_1=0, \text{ etc.}$$

These equations (in which p_1, p_2, etc., and π_1, π_2, etc., are interchangeable determine $p_1, p_2, \dots\dots\dots, p_n$ in terms of $\pi_1, \pi_2, \dots \pi_n$, and *vice versa*.

By means of this and the preceding example the values of the following symmetric functions may be found in terms of the co-efficients :—

$$\Sigma \frac{\alpha^{n-1}}{f'(\alpha)},\ \Sigma \frac{\alpha^n}{f'(\alpha)},\ \Sigma \frac{\alpha^{n+1}}{f'(\alpha)}, \text{ etc.}$$

3. Express π_r by the sums of the powers of the roots.

Representing by $\frac{1}{u}$ the product $(1-\alpha_1 y)(1-\alpha_2 y)\dots(1-\alpha_n y)$, and differentiating, we find

$$\frac{1}{u}\frac{du}{dy} = \Sigma \frac{\alpha}{1-\alpha y} = s_1+s_2 y+s_3 y^2+\dots;$$

also
$$u = 1+\pi_1 y+\pi_2 y^2+\dots\dots$$

We have, therefore,

$$(1+\pi_1 y+\pi_2 y^2+\dots)(s_1+s_2 y+s_3 y^2+\dots) \equiv \pi_1+2\pi_2 y+3\pi_3 y^2+\dots$$

Now, comparing the several co-efficients of the different powers of y, we have a number of equations by means of which the sums of the homogeneous products $\pi_1, \pi_2, \pi_3, \dots$, may be expressed in terms of s_1, s_2, s_3, etc.

4. Prove the following formula for calculating the sums of the homogeneous products in terms of the co-efficients :—

$$\frac{ds_{r+i}}{dp_r} = -(r+i)\pi_i.$$

Differentiate both sides of equation (1) in Art. 80, and introduce π_1, π_2, etc. by the equation of Ex. 2.

CHAPTER IX

LIMITS OF THE ROOTS OF EQUATIONS

84. Definition of Limits. In attempting to discover the real roots of numerical equations, it is in the first place advantageous to narrow the region within which they must be sought. We here take up the inquiry referred to in the observation at the end of Art. 4, and proceed to prove certain propositions relative to the limits of the real roots of equations.

A *superior limit* of the positive roots is any greater positive number than the greatest of these roots; an *inferior limit* of the positive roots is any smaller positive number than the smallest of them. A superior limit of the negative roots is any greater negative number than the greatest of them; an inferior limit of the negative roots is any smaller negative number than the smallest; the greatest negative number meaning here the number nearest to $-\infty$.

When we have found limits within which all the real roots of an equation lie, the next step towards the solution of the equation is to discover the intervals in which the separate roots are situated. The principal methods in use for this latter purpose will form the subject of the next chapter.

The following Propositions all relate to the superior limits of the positive roots; to which, as will be subsequently proved, the determination of inferior limits and limits of the negative roots can be immediately reduced.

85. Proposition I. *In any equation*

$$x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots+p_{n-1}x+p_n=0,$$

if the first negative term be $-p_rx^{n-r}$, *and if the greatest negative coefficient be* $-p_k$, then $\sqrt[r]{p_k}+1$ *is a superior limit of the positive roots.*

Any value of x which makes

$$x^n>p_k(x^{n-r}+x^{n-r-1}+\ldots+x+1)>p_k\frac{x^{n-r+1}-1}{x-1}$$

will, *a fortiori*, make $f(x)$ positive.

Now, taking x greater than unity, this inequality is satisfied by the following:—

$$x^n>p_k\frac{x^{n-r+1}}{x-1},$$

or $$x^{n+1}-x^n > p_k x^{n-r+1},$$

or $$x^{r-1}(x-1) > p_k,$$

which inequality again is satisfied by the following :—

$$(x-1)^{r-1}(x-1) = \text{or} > p_k,$$

since plainly $$x^{r-1} > (x-1)^{r-1}$$

We have, therefore, finally

$$(x-1)^r = \text{or} > p_k,$$

or $$x = \text{or} > 1+\sqrt[r]{p_k}.$$

86. Proposition II. *If in any equation each negative coefficient be taken positively, and divided by the sum of all the positive coefficients which precede it, the greatest quotient thus formed increased by unity is a superior limit of the positive roots.*

Let the equation be

$$a_0x^n+a_1x^{n-1}+a_2x^{n-2}-a_3x^{n-3}+\ldots-a_rx^{n-r}+\ldots+a_n=0,$$

in which, in order to fix our ideas, we regard the fourth coefficient as negative, and we consider also a negative coefficient in general, *viz.*, $-a_r$.

Let each positive term in this equation be transformed by means of the formula

$$a_mx^m = a_m(x-1)(x^{m-1}+x^{m-2}+\ldots+x+1)+a_m,$$

which is derived at once from

$$\frac{x^m-1}{x-1} = x^{m-1}+x^{m-2}+\ldots+x+1\ ;$$

the negative terms remaining unchanged.

The polynomial $f(x)$ becomes then, the horizontal lines of the following corresponding to the successive terms of $f(x)$:—

$$\begin{array}{r}
a_0(x-1)x^{n-1}+a_0(x-1)x^{n-2}+a_0(x-1)x^{n-3}+\ldots+a_0(x-1)x^{n-r}+\ldots+a_0,\\
+a_1(x-1)x^{n-2}+a_1(x-1)x^{n-3}+\ldots+a_1(x-1)x^{n-r}+\ldots+a_1,\\
+a_2(x-1)x^{n-3}+\ldots+a_2(x-1)x^{n-r}+\ldots+a_2,\\
-a_3x^{n-3},\qquad\qquad\qquad\qquad\qquad\quad\\
+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
-a_rx^{n-r},\qquad\quad\\
+\ldots\ldots\ldots\\
+a_n.
\end{array}$$

We now regard the vertical columns of this expression as successive terms in the polynomial; the successive coefficients of x^{n-1}, x^{n-2}, etc., being

$$a_0(x-1),\ (a_0+a_1)(x-1),\ (a_0+a_1+a_2)(x-1)-a_3, \text{ etc.}$$

Any value of x greater than unity is sufficient to make positive every term in which no negative coefficient a_3, a_r, etc., occurs. To make the latter terms positive, we must have

$$(a_0+a_1+a_2)(x-1)>a_3,$$

$$(a_0+a_1+a_2+\ldots+a_{r-1})(x-1)>a_r, \text{ etc.}$$

Hence

$$x>\frac{a_3}{a_0+a_1+a_2}+1, \ldots x>\frac{a_r}{a_0+a_1+a_2+a_4+\ldots+a_{r-1}}+1, \text{ etc.}$$

And to ensure every term being made positive, we must take the value of the greatest of the quantities found in this way. Such a value of x, therefore, is a superior limit of the positive roots.

87. Practical Applications. The propositions in the two preceding Articles furnish the most convenient *general* methods of finding in practice tolerably close limits of the roots. Sometimes one of the propositions will give the closer limit, sometimes the other. It is well, therefore, to apply both methods, and take the smaller limit. Prop. I will usually be found the more advantageous when the first negative coefficient is preceded by several positive coefficients, so that r is large; and Prop. II when large positive coefficients occur before the first large negative coefficient. In general, Prop. II will give the closer limit. We speak of the integer next above the numerical value given by either proposition as the limit.

Examples

1. Find a superior limit of the positive roots of the equation

$$x^4-5x^3+40x^2-8x+23=0.$$

Prop. I gives $8+1$, or 9, as limit.

Prop. II gives $\frac{5}{1}+1$, or 6. Hence 6 is a superior limit.

2. Find a superior limit of the positive roots of the equation

$$x^5+3x^4+x^3-8x^2-51x+18=0.$$

Prop. I gives $\sqrt[3]{51}+1$; and 5 is, therefore, a limit.

Prop. II gives $\frac{51}{1+3+1}+1$, and 12 is a limit.

In this case Prop. I gives the closer limit.

3. Find a superior limit of the positive roots of

$$x^7+4x^6-3x^5+5x^4-9x^3-11x^2+6x-8=0.$$

Of the fractions

$$\frac{3}{1+4}, \frac{9}{1+4+5}, \frac{11}{1+4+5}, \frac{8}{1+4+5+6},$$

the third is the greatest, and Prop. II gives the limit 3, Prop. I gives 5.

4. Find a superior limit of the positive roots of

$$x^8+20x^7+4x^6-11x^5-120x^4+13x-25=0.$$

[*Ans.* Both methods give the limit 6.

5. Find a superior limit of the positive roots of

$$4x^5-8x^4+22x^3+98x^2-73x+5=0.$$

[*Ans.* Prop. I gives 20. Prop. II gives 3

It is usually possible to determine by inspection a limit closer than that given by either of the preceding propositions. This method consists in arranging the terms of an equation in groups having a positive term first, and then observing what is the lowest integer value of x which will have the effect of rendering each group positive. The form of the equation will suggest the arrangement in any particular case.

6. The equation of Ex. 2 can be arranged as follows :-

$$x^2(x^3-8)+x(3x^3-51)+x^3+18=0.$$

$x=3$, or any greater number, renders each group positive ; hence 3 is a superior limit.

7. The equation of Ex. 4 may be arranged thus :

$$x^5(x^3-11)+20x^4(x^3-6)+4x^6+13x-25=0.$$

$x=3$, or any greater number, renders each group positive ; hence 3 is a limit.

8. Find a superior limit of the roots of the equation

$$x^4-4x^3+33x^2-2x+18=0.$$

This can be arranged in the form

$$x^2(x^2-4x+5)+28x(x-\tfrac{1}{14})+18=0.$$

Now the trinomial x^2-4x+5, having imaginary roots, is positive for all values of x (Art. 12). Hence $x=1$ is a superior limit.

The introduction in this way of a quadratic whose roots are imaginary, or of one with equal roots, will often be found useful.

9. Find a superior limit of the roots of the equation

$$5x^5-7x^4-10x^3-23x^2-90x-317=0$$

In examples of this kind it is convenient to distribute the highest power of x among the negative terms. Here the equation may be written

$$x^4(x-7)+x^3(x^2-10)+x^2(x^3-23)+x(x^4-90)+x^5-317=0,$$

so that 7 is evidently a superior limit of the roots. In this case the general methods give a very high limit.

10. Find a superior limit of the roots of the equation

$$x^4-x^3-2x^2-4x-24=0.$$

When there are several negative terms and the coefficient of the highest term unity, it is convenient to multiply the whole equation by such a number as will enable us to distribute the highest term among the negative terms. Here, multiplying by 4, we can write the equation as follows :—

$$x^3(x-4)+x^2(x^2-8)+x(x^3-16)+x^4-96=0,$$

and 4 is a superior limit. The general methods give 25.

88. Proposition III. *Any number which renders positive the*

polynomial $f(x)$ *and all its derived functions* $f_1(x), f_2(x), \ldots, f_n(x)$ *is a superior limit of the positive roots of the equation* $f(x)=0$.

This method of finding limits is due to Newton. It is much more laborious in its application than either of the preceding methods ; but it has the advantage of giving always very close limits ; and in the case of an equation all whose roots are real the limit found in this way is, as will be subsequently proved, the next integer above the greatest positive root.

To prove the proposition, let the roots of the equation $f(x)=0$ be diminished by h ; then $x-h=y$, and

$$f(y+h)=f(h)+f_1(h)y+\frac{f_2(h)}{1.2}y^2+\ldots+\frac{f_n(h)}{1.2\ldots n}y^n.$$

If now h be such as to make all the coefficients

$$f(h), f_1(h), f_2(h), \ldots, f_n(h)$$

positive, the equation in y cannot have a positive root ; that is to say, the equation in x has no root greater than h ; hence h is a superior limit of the positive roots.

Example

$$f(x) \equiv x^4-2x^3-3x^2-15x-3.$$

In applying Newton's method of finding limits to any example the general mode of procedure is as follows:—Take the smallest integer number which renders $f_{n-1}(x)$ positive ; and proceeding upwards in order to $f_1(x)$, try the effect of substituting this number for x in the other functions of the series. When any function is reached which becomes negative for the integer in question, increase the integer successively by units, till it makes that function positive ; and then proceed with the new integer as before, increasing it again if another function in the series should become negative ; and so on, till an integer is reached which renders all the functions in the series positive. In the present example the series of functions is

$$f(x)=x^4-2x^3-3x^2-15x-3,$$
$$f_1(x)=4x^3-6x^2-6x-15,$$
$$\tfrac{1}{2}f_2(x)=6x^2-6x-3,$$
$$\tfrac{1}{6}f_3(x)=4x-2,$$
$$\tfrac{1}{24}f_4(x)=1.$$

Here $x=1$ makes $f_3(x)$ positive. We try then the effect of the substitution $x=1$ in $f_2(x)$. It makes $f_2(x)$ negative. Increase by 1 ; and $x=2$ makes $f_2(x)$ positive. Try the effect of $x=2$ in $f_1(x)$; it gives a negative result. Increase by 1 ; and $x=3$ makes $f_1(x)$ positive. Proceeding upwards, the substitution $x=3$ makes $f(x)$ negative ; and increasing again by unity, we find that $x=4$ makes $f(x)$ positive. Hence 4 is the superior limit required.

It is assumed in this mode of the applying Newton's rule, that when any number makes all the derived functions up to a certain stage positive, any higher number will also make them positive ; so that there is no occasion to try the effect of the higher number on the functions lower down in the series. This is evident from the equation

$$\varphi(a+h)=\varphi(a)+\varphi'(a)h+\varphi''(a)\frac{h^2}{1.2}+\ldots\ldots$$

[taking $\varphi(x)$ to represent any function in the series, and using the common notation for derived functions], which shows that if $\varphi(a)$, $\varphi'(a)$, $\varphi''(a)$,...are all positive, and h also positive, $\varphi(a+h)$ must be positive.

It may be observed that one advantage of Newton's method is that often, as in the present instance, it gives us a knowledge of the two successive integers between which the highest root lies. Thus in the present example, since $f(x)$ is negative for $x=3$, and positive for $x=4$, we know that the greatest root of the equation lies between 3 and 4.

89. Inferior Limits, and Limits of the Negative Roots. To find an inferior limit of the positive roots, the equation must be first transformed by the substitution $x=\frac{1}{y}$. Find then a superior limit h of the positive roots of the equation in y. The reciprocal of this, *viz.*, $\frac{1}{h}$, will be the required inferior limit ; for since

$$y<h,\ \frac{1}{y}>\frac{1}{h},\ i.e.,\ x>\frac{1}{h}.$$

To find limits of the negative roots, we have only to transform the equation by the substitution $x=-y$. This transformation changes the negative into positive roots. Let the superior and inferior limits of the positive roots of the equation in y be h and h'. Then $-h$ and $-h'$ are the limits of the negative roots of the proposed equation.

90. Limiting Equations. *If all the real roots of the equation $f'(x)=0$ could be found, it would be possible to determine the number of real roots of the equation $f(x)=0$.*

To prove this, let the real roots of $f'(x)=0$ be, in ascending order of magnitude, α', β', γ',..., λ' ; and let the following series of values be substituted for x in $f(x)$:—

$$-\infty,\ \alpha',\ \beta',\ \gamma',\ldots,\ \lambda',+\infty.$$

When any successive two of these quantities give results with different signs there is a root of $f(x)=0$ between them ; and by the Cor., Art. 71, there is only one ; and when they give results with the same sign, there is, by the same Cor., no root between them. Thus each change of sign in the results of the successive substitutions proves the existence of one real root of the proposed equation.

If all the roots of $f(x)=0$ are real, it is evident, by the theorem of Art. 71, that all the roots of $f'(x)=0$ are also real, and that they lie one by one between each adjacent pair of the roots of $f(x)=0$. In the same case, and by the same theorem, it follows that the roots of $f''(x)=0$, and of all the successive derived functions, are real also ; and the roots of any function lie severally between each adjacent pair of the roots of the function from which it is immediately derived.

Equations of this kind, which are one degree below the degree of any proposed equation. and whose roots lie severally between each adjacent pair of the roots of the proposed, are called *limiting equations*.

It is evident that in the application of Newton's method of finding limits of the roots, when the roots of $f(x)=0$ are all real, in proceeding according t the method explained in Art. 88, the function $f(x)$ is itself the last which will be rendered positive, and, therefore, the superior limit arrived at is the integer next above the greatest root.

Examples

1. Prove that any derived equation $f_m(x)=0$ cannot have more imaginary roots, but may have more real roots, than the equation $f(x)=0$ from which it is derived.

From this it follows that, if any of the derived functions be found to have imaginary roots, the same number at least of imaginary roots must enter the original equation.

2. Apply the method of Art. 90 to determine the conditions that the equation

$$x^3-qx+r=0$$

should have all its roots real.

3. Determine by the same method the nature of the roots of the equation

$$x^n-nqx+(n-1)r=0.$$

[*Ans.* When n is even, the equation has two real roots or none, according as

$$q^n > \text{ or } < r^{n-1}.$$

When n is odd, the equation has three real roots or one, according as

$$q^n > \text{ or } < r^{n-1}.$$

4. The equation $x^n(x-1)^n=0$ has all its roots real; hences show, by forming the n^{th} derived function, that the following equation has all its roots real and unequal, and situated between 0 and 1 :—

$$x^n-n\frac{n}{2n}x^{n-1}+\frac{n(n-1)}{1.2}\frac{n(n-1)}{2n(2n-1)}x^{n-2}-\text{etc.}=0.$$

5. Show similarly by forming the n^{th} derived of $(x^2-1)^n$ that the following equation has all its roots real and unequal, and situated between -1 and 1 :

$$x^n-n\frac{n(n-1)}{2n(2n-1)}x^{n-2}+\frac{n(n-1)}{1.2}\frac{n(n-1)(n-2)(n-3)}{2n(2n-1)(2n-2)(2n-3)}x^{n-4}-\text{etc.}=0.$$

6. If any two of the quantities, l, m, n in the following equation be put equal to zero, show that the quadratic to which the equation then reduces is a limiting equation ; and hence prove that the roots of the proposed are all real :—

$$(x-a)(x-b)(x-c)-l^2(x-a)-m^2(x-b)-n^2(x-c)-2lmn=0.$$

7. Discuss the nature of the roots of the equation

$$x^4+4x^3-2x^2-12x+p=0,$$

according to the different values of p.

Applying Art. 90. When p is less than -7, two roots are real and two imaginary ; when p lies between -7 and 9, all the roots are real ; and when p is greater than 9, the roots are all imaginary. The equation has two equal roots when $p=-7$, and two pairs of equal roots when $p=9$.

CHAPTER X

SEPARATION OF THE ROOTS OF EQUATIONS

91. By the methods of the preceding chapter we are enabled to find limits between which all the real roots of any numerical equation lie. Before proceeding to the actual approximation to any particular root, it is necessary to separate the interval in which it is situated from the intervals which contain the remaining roots. The present chapter will be occupied with certain theorems whose object is to determine the number of real roots between any two arbitrarily assumed values of the variable. It is plain that if this object can be effected, it will then be possible to tell not only the total number of real roots, but also the limits within which the roots separately lie.

The theorems given for this purpose by Fourier and Budan, although different in statement, are identical in principle. For purposes of exposition Fourier's statement is the more convenient, while with a view to practical application the statement of Budan will be found superior. The theorem of Sturm, although more laborious in practice, has the advantage over the preceding that it is unfailing in its application, giving always the exact number of real roots situated between any two proposed quantities; whereas the theorem of Fourier and Budan gives only a certain limit which the number of real roots in the proposed interval cannot exceed.

92. Theorem of Fourier and Budan. *Let two numbers a and b, of which a is the less, be substituted in the series formed by $f(x)$ and its successive derived functions, viz.,*

$$f(x), f_1(x), f_2(x), \ldots\ldots, f_n(x);$$

the number of real roots which lie between a and b cannot be greater than the excess of the number of changes of sign in the series when a is substituted for x, over the number of changes when b is substituted for x; and when the number of real roots in the interval falls short of that difference, it will be by an even number.

This is the form in which Fourier states the theorem.

It is to be understood here, as elsewhere, that, when we speak of two numbers a and b, of which a is the less, one or both of them may be negative, and what is meant is that a is nearer than b to $-\infty$

We proceed to examine the changes which may occur among the signs of the functions in the above series, the value of x being supposed to increase continuously from a to b. The following different cases can arise :—

(1) The value of x may pass through a single root of the equation $f(x)=0$.

(2) It may pass through a root occurring r times in $f(x)=0$.

(3) It may pass through a root of one of the auxiliary functions $f_m(x)=0$, this root not occurring in either $f_{m-1}(x)=0$ or $f_{m+1}(x)=0$.

(4) It may pass through a root occurring r times in $f_m(x)=0$, and not occurring in $f_{m-1}(x)=0$.

In what follows the symbol x is omitted after f for convenience.

(1) In the first case it is evident, from Art. 75, that in passing through a root of the equation $f(x)=0$, one change of sign is lost ; for f and f_1 have unlike signs immediately before, and like signs immediately after, the passage through the root.

(2) In the second case, in passing through an r-multiple root of $f(x)=0$, it is evident that r changes of sign are lost ; for, by Art. 76, immediately before the passage the series of functions

$$f, f_1, f_2, \ldots \ldots, f_{r-1}, f_r$$

have signs alternately $+$ and $-$, or $-$ and $+$, and immediately after the passage have all the same sign as f_r.

(3) In the third case, the root of $f_m(x)=0$ must give to f_{m-1} and f_{m+1} either like signs or unlike signs. Suppose it to give like signs ; then in passing through the root two changes of sign are lost, for before the passage the sign of f_m is different from these like signs, and after the passage it is the same (Art. 76). Suppose it to give unlike signs ; then no change of sign is lost, for before the passage the signs of f_{m-1}, f_m, f_{m+1} must be either $+\ +$, or $-\ -\ +$ and after the passage these become $+\ -\ -$, and $-\ +\ +$. On the whole, therefore, we conclude that no variation of sign can be gained, but two variations may be lost, on the passage through a root of $f_m(x)=0$.

(4) In the fourth case x passes through a value (let us say α) which causes not only f_m, but also $f_{m+1}, f_{m+2}, \ldots\ldots, f_{m+r-1}$ to vanish. It is evident from the theorem of Art. 76 that during the passage a number of changes of sign will always be lost. The definite number may be collected by considering the series of functions

$$f_{m-1}, f_m, f_{m+1}, \ldots\ldots, f_{m+r-1}, f_{m+r}.$$

The following results are easily established :—

(*a*) When $f_{m-1}(\alpha)$ and $f_{m+r}(\alpha)$ have like signs ;

If r be even, r changes are lost.

If r be odd, $r+1$ changes are lost.

(*b*) When $f_{m-1}(\alpha)$ and $f_{m+r}(\alpha)$ have unlike signs :

If r be even, r changes are lost.

If r be odd, $r-1$ changes are lost.

We conclude, therefore, on the whole, that an even number of changes is lost during the passage through an r-multiple root of $f_m(x)$.

It will be observed that (1) is a particular case of (2), and (3) of (4), *i.e.*, when $r=1$. Since, however, the cases (1) and (3) are those of ordinary occurrence, it is well to give them a separate classification.

Reviewing the above proof, we conclude that as x increases from a to b no change of sign can be gained ; that for each passage through a single root of $f(x)=0$ one change is lost ; and that under no circumstances except a passage through a root of $f(x)=0$ can an odd number of changes be lost. Hence the number of changes l st during the whole variation of x from a to b must be either equal to the number of real roots of $f(x)=0$ in the interval, or must exceed it by an even number. The theorem is, therefore, proved.

93. Application of the Theorem. The form in which the theorem has been stated by Budan is, as has been already observed, more convenient for practical purposes than that just given. It is as follows :—*Let the roots of an equation $f(x)=0$ be diminished, first by a and then by b, where a and b are any two numbers of which a is the less ; then the number of real roots between a and b cannot be greater than the excess of the number of changes of sign in the first transformed equation over the number in the second.*

This is evidently included in Fourier's statement, for the two transformed equations are (see Art. 33)—

$$f(a)+f_1(a)y+\frac{f_2(a)}{1.2}y^2+\ldots\ldots+\frac{f_n(a)}{1.2\ldots\ldots n}y^n=0,$$

$$f(b)+f_1(b)y+\frac{f_2(b)}{1.2}y^2+\ldots\ldots+\frac{f_n(b)}{1.2\ldots\ldots n}y^n=0\ ;$$

from which, assuming the results of the last Article, the above proposition is manifest.

The reason why the theorem in this form is convenient in practice is, that we can apply the expeditious method of diminishing the roots given in Art. 33.

Examples

1. Find the situations of the roots of the equation

$$x^5-3x^4-24x^3+95x^2-46x-101=0.$$

We shall examine this function for values of p between the intervals

$$-10, \qquad -1, \qquad 0, \qquad 1, \qquad 10\,;$$

these numbers being assumed on account of the facility of calculation. Diminution of the roots by 1 gives the following series of coefficients of the transformed equation :—

$$1, \qquad 2, \qquad -26, \qquad 15, \qquad 65 \qquad -78.$$

In diminishing the roots by 10, it is apparent at the very outset of the calculation that the signs of the co-efficients of the transformed equation will be all positive ; so that there is no occasion to complete the calculation in this case.

In diminishing the roots by -10 and -1, it is convenient to change the alternate signs of the equation, and diminish the roots by $+10$ and $+1$; and then in the result change the alternate signs again. The coefficients of the transformed equation when the roots are diminished by -1 are

$$1, \qquad -8, \qquad -2, \qquad 139, \qquad -291, \qquad 60.$$

In diminishing by -10 we observe in the course of the operation, as before, that the signs will be all positive in the result, *i.e.*, when the alternate signs are changed they will be alternately positive and negative.

Hence we have the following scheme :—

$$\begin{array}{rl}
(-10) & + - + - + -, \\
(-1) & + - - + - +, \\
(0) & + - - + - -, \text{ the equation itself} \\
(1) & + + - + + -, \\
(10) & + + + + + +.
\end{array}$$

These signs are the signs taken by $f(x)$ and the several derived functions f_1, f_2, f_3, f_4, f_5 on the substitution of the proposed numbers; but it is to be observed that they are here written, not in the order of Art. 92 but in the reverse order, *viz.*, $f_5, f_4, f_3, f_2, f_1, f$.

From these we draw the following conclusions :—All the real roots must lie between -10 and $+10$; one real root lies between -10 and -1, since one change of sign in lost ; one real root lies between -1 and 0, since one change of sign is lost ; no real root lies between 0 and 1 ; and between 1 and 10, since three changes of sign are lost, there is at least one real root ; but we are left in doubt as to the nature of the other two roots : whether they are imaginary, or whether there are three real roots between 1 and 10.

We might proceed to examine, by further transformations, the interval between 1 and 10 more closely, in order to determine the nature of the two doubtful roots ; but it is evident that the calculations for this purpose might, if the roots were nearly equal, become very laborious. This is the weak side of the theorem of Fourier and Budan. Both writers have attempted to supply this defect, and have given methods of determining the nature of the roots in doubtful intervals ; but as these methods are complicated, we do not stop to explain them ; the more especially as the theorem of Sturm effects fully the purposes for which the supplementary methods of Fourier and Budan were invented.

2. Analyse the equation of Ex. 1, p. 81, *viz.*,

$$x^3+x^2-2x-1=0.$$

The roots of this are all real, and lie between -2 and 2 (*see* Ex. 5, p. 81). Whenever the roots of an equation are all real, the signs of Fourier's functions determine the exact number of real roots between any two proposed integers. We obtain the following result :— The roots lie in the intervals

$$(-2, -1)\ ;\ (-1, 0)\ ;\ (1, 2).$$

3. Analyse the equation of Ex. 3, p. 81, *viz.*,

$$x^5+x^4-4x^3-3x^2+3x+1=0.$$

[*Ans.* Two roots in the interval $(-2, -1)$, and one root in each of the intervals $(-1, 0)$; $(0, 1)$; $(1, 2)$.

4. Analyse the equation

$$x^4-80x^3+1998x^2-14937x+5000=0.$$

The equation can have no negative roots. Diminish the roots by 10 several times in succession till the signs of the coefficients become all positive. We obtain the following result :—

(0)	+	−	+	−	+,
(10)	+	−	+	+	−,
(20)	+	0	−	+	+,
(30)	+	+	+	−	+,
(40)	+	+	+	+	+.

Thus, there is one root between 0 and 10, and one between 10 and 20 ; no root between 20 and 30. Between 30 and 40 either there are two real roots, or there is an indication of a pair of imaginary roots. That the former is the case will appear by diminishing the roots of the third transformed equation by units. This process will separate the roots, which will be found to lie between (2, 3) and (4, 5); so that the proposed equation has a third real root in the interval (32, 33), and a fourth in the interval (34, 35).

94. Application of the Theorem to Imaginary Roots. Since there exist only n changes of sign to be lost in the passage of x from $-\infty$ to $+\infty$, if we have any reason for knowing that a pair of changes is lost during the passage of x through an interval which includes no real root of the equation, we may be assured of the existence of a pair of imaginary roots. Circumstances of this nature will arise in the application of Fourier's theorem when any of the transformed equations contain vanishing coefficients. For we can assign by the principle of Art. 76 the proper sign to this coefficient, corresponding to values of x immediately before and immediately after that value which causes the coefficient to vanish ; the whole interval being so small that it may be supposed not to include any root of the equation $f(x)=0$.

Examples

1. Analyse the equation

$$f(x)\equiv x^4-4x^3-3x+23=0.$$

We shall examine this function between the intervals 0, 1, 10 The transformed equations are

$$\tfrac{1}{24}f_4(0)x^4+\tfrac{1}{6}f_3(0)x^3+\tfrac{1}{2}f_2(0)x^2+f_1(0)x+f(0) \quad =0,$$
$$\tfrac{1}{24}f_4(1)x^4+\tfrac{1}{6}f_3(1)x^3+\tfrac{1}{2}f_2(1)x^2+f_1(1)x+f(1) \quad =0,$$
$$\tfrac{1}{24}f_4(10)x^4+\tfrac{1}{6}f_3(10)x^3+\tfrac{1}{2}f_2(10)x^2+f_1(10)x+f(10)=0,$$

the first of these being the proposed equation itself.

Making the calculations by the method of the preceding Article, we find that the coefficient $f_3(1)=0$, and we have the following scheme :—

$$\begin{array}{lccccc} (0) & + & - & 0 & - & +, \\ (1) & + & 0 & - & - & +, \\ (10) & + & + & + & + & +. \end{array}$$

We may now replace each of the rows containing a zero coefficient by two, the first corresponding to a value a little less, and the second to a value a little greater, than that which gives the zero coefficients, the signs being determined by the principle established in Art. 76. It must be remembered that in the above scheme the signs representing the derived functions are written in the reverse order to that of the Article referred to. The scheme will then stand as follows, h being used to represent a very small positive quantity :—

$$\begin{array}{llccccc} (0) & -h & + & - & + & - & +, \\ & +h & + & - & - & - & +, \\ (1) \begin{cases} \\ \\ \end{cases} & 1-h & + & - & - & - & +, \\ & 1+h & + & + & - & - & +, \\ (10) & & + & + & + & + & +, \end{array}$$

where the signs corresponding to $-h$ and $+h$ are determined by the condition that the sign of the coefficient which is zero when $x=0$ must, when $x=-h$, be different from that next to it on the left-hand side, and when $x=+h$ those signs must be the same. The signs corresponding to $1-h$ and $1+h$ are determined in a similar manner.

Now since a pair of changes is lost in the interval $(-h, +h)$, and since the equation has no real root between $-h$ and $+h$, we have proved the existence of a pair of imaginary roots. Two changes of sign are lost between $1+h$ and 10, so that this interval either includes a pair of real roots, or presents an indication of a pair of imaginary roots. Which of these is the case remains still doubtful.

2. If several coefficients vanish, we may be able to establish the existence of several pairs of imaginary roots. This will appear from the following exmaple :—

$$x^6-1=0.$$

The signs corresponding to $-h$ and $+h$ are, by the theorem of Art. 76,

$$\begin{array}{lcccccc} (-h) & + & - & + & - & + & - & -, \\ (+h) & + & + & + & + & + & + & -. \end{array}$$

Hence, since no root exists between $-h$ and $+h$, and since 4 changes of sign are lost in passing from a value very little less than 0 to one very little greater, we are assured of the existence of two pairs of imaginary roots. The other two roots are in this case plainly real (*see* Art. 14).

The number of imaginary roots in any binomial equation can be determined in this way.

3. Find the character of the roots of the equation

$$x^8+10x^3+x-4=0.$$

In passing from a small negative to a small positive value of x we obtain the following series of signs :—

$$\begin{array}{rccccccccc} (-h) & + & - & + & - & + & + & - & + & -, \\ (0). & + & 0 & 0 & 0 & 0 & + & 0 & + & -, \\ (+h) & + & + & + & + & + & + & + & + & -. \end{array}$$

Since six changes of sign are here lost, there are six imaginary roots. The remaining two roots are, by Art. 14, real : one positive, and the other negative. The negative root lies between -2 and -1, and the positive between 0 and 1.

4. Analyse completely the equation

$$x^4-3x^2-x+1=0.$$

There are two imaginary roots. Whenever, as in the present instance, the roots are comprised within small limits, it is convenient to diminish by successive units. In this way we find here a root between 0 and 1, and another between 1 and 2. Proceeding to negative roots, we find on diminishing by -1 that -1 is itself a root, and writing down the signs corresponding to a value a little greater than -1, we observe an indication of a second negative root between -1 and 0.

5. Analyse the equation

$$x^5+x^4+x^2-25x-36=0.$$

There are two imaginary roots ; one real positive root between 2 and 3; and two real negative roots in the intervals $(-3, -2)$, $(-2, -1)$.

95. Corollaries from the Theorem of Fourier and Budan. The method of detecting the existence of imaginary roots explained in the preceding Article is called *The Rule of the Double Sign.* A similar rule, due to *De Gua,* was in use before the discovery of Fourier's theorem. This rule and Descartes' *Rule of Signs* are immediate corollaries from the theorem, as we proceed to show.

Cor. 1. *De Gua's Rule for finding imaginary Roots.*

The rule may be stated generally as follows :—*When 2m successive terms of an equation are absent, the equation has 2m imaginary roots ; and when 2m+1 successive terms are absent, the equation has 2m+2, or 2m imaginary roots, according as the two terms between which the deficiency occurs have like or unlike signs.* This follows, as in case (4), Art. 92, by examining the number of changes of sign lost during the passage of x from a small negative value $-h$ to a small positive value h.

Cor. 2. *Descartes' Rule of Signs.*

When 0 is substituted for x in the series of functions $f_n(x), f_{n-1}(x), \ldots, f_2(x), f_1(x), f(x)$, the signs are the same as the signs of the coefficients $a_0, a_1, a_2, \ldots, a_{n-1}, a_n$, of the proposed equation ;

and when $+\infty$ is substituted the signs are all positive. Fourier's theorem asserts that the number of roots between these limits, *viz.*, the number of positive roots, cannot exceed the number of variations lost during the passage from 0 to $+\infty$, that is the number of changes of sign in the series $a_0, a_1, a_2, \ldots, a_n$. This is Descartes' rule for positive roots ; and the similar rule for negative roots follows in the usual way by changing the negative into positive roots.

Cor. 3. *Newton's Method of finding Limits.*

When a number h has been found which renders positive each of the functions $f_n(x), f_{n-1}(x), \ldots, f_2(x), f_1(x), f(x)$; since $+\infty$ also renders each of them positive, it follows from Fourier's theorem that there can be no root between h and $+\infty$, that is to say, h is a superior limit of the positive roots ; and this is Newton's proposition (Art. 88)

96. Sturm's Theorem. We have already shown (Art. 74) that it is possible by performing the common algebraical operation of finding the greatest common measure of a polynomial $f(x)$ and its first derived polynomial to find the equal roots of the equation $f(x)=0$. Sturm has employed the same operation for the formation of the auxiliary functions which enter into his method of separating the roots of an equation.

Let the process of finding the greatest common measure of $f(x)$ and its first derived be performed. The successive remainders will go on diminishing in degree till we reach finally either one which divides that immediately preceding without remainder, or one which does not contain the variable at all, *i.e.*, which is numerical. The former is, as we have already seen, the case of equal roots. The latter is the case where no equal roots exist. It is convenient to divide the discussion of Sturm's theorem into these two cases. We shall in the present Article consider the case where no equal roots exist ; and proceed in the next Article to the case of equal roots. The performance of the operation itself will of course disclose the class to which any particular example is to be referred.

The auxiliary functions employed by Sturm are not the remainders as they present themselves in the calculation, but the remainders *with their signs changed*. In finding the greatest common measure of two expressions it is indifferent whether the signs of the remainders are changed or not : in the formation of Sturm's auxiliary functions the change is essential. We shall suppose, therefore, in what follows that the sign of each remainder is changed before it becomes the next divisor.

Confining our attention for the present to the case where no equal roots exist, Sturm's theorem may be stated as follows :—

Theorem. *Let any two real quantities a and b be substituted for x in the series of $n+1$ functions*

$$f(x), f_1(x), f_2(x), f_3(x), \ldots, f_{n-1}(x), f_n(x),$$

consisting of the given polynomial $f(x)$, its first derived $f_1(x)$, and the successive remainders (with their signs changed) in the process of finding the greatest common measure of $f(x)$ and $f_1(x)$; then the difference between the number of changes of sign in the series when a is substituted for x and the number when b is substituted for x expresses exactly the number of real roots of the equation $f(x)=0$ between a and b.

The mode of formation of Sturm's functions supplies the following series of equations, in which $q_1, q_2, \ldots, q_{n-1}$ represent the successive quotients in the operation :—

$$\left.\begin{array}{l} f(x) \quad = q_1 f_1(x) - f_2(x), \\ f_1(x) \quad = q_2 f_2(x) - f_3(x), \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ f_{r-1}(x) = q_r f_r(x) - f_{r+1}(x), \\ \\ f_{n-2}(x) = q_{n-1} f_{n-1}(x) - f_n(x). \end{array}\right\} \quad \ldots(1)$$

These equations involve the theory of the method of finding the greatest common measure ; for it follows from the first equation that if $f(x)$ and $f_1(x)$ have a common factor, this must be a factor in $f_2(x)$; and from the second equation it follows, by like reasoning, that the same factor must occur in $f_3(x)$; and so on, till we come finally to the last remainder, which, when $f(x)$ and $f_1(x)$ have common factors, will be a polynomial consisting of these factors. In the present Article, where we suppose the given polynomial and its first derived to have no common factor, the last remainder $f_n(x)$ is numerical. It is essential for the proof of the theorem to observe also, that in the case now under consideration no two consecutive functions in the series can have a common factor ; for if they had we could, by reasoning similar to the above, show from the equations that this factor must exist also in $f(x)$ and $f_1(x)$; and such, according to our hypothesis, is not here the case. In examining, therefore, what changes of sign can take place in the series during the passage of x from a to b, we may exclude the case of two consecutive functions vanishing for the same value of the variable ; and the different cases in which any change of sign can take place are the following :—

(1) when x passes through a root of the proposed equation $f(x)=0$,

(2) when x passes through a value which causes one of the auxiliary functions $f_1, f_2, \ldots, f_{n-1}$ to vanish,

(3) when x passes through a value which causes two or more of the series $f, f_1, \ldots, f_{n-1}$ to vanish together, no two of the vanishing functions, however, being consecutive.

(1) When x passes through a root of $f(x)=0$, it follows from Art. 75 that one change of sign is lost, since immediately before the passage $f(x)$ and $f_1(x)$ have unlike signs, and immediately after the passage they have like signs.

(2) Suppose x to take a value α which satisfies the equation $f_r(x)=0$. From the equation

$$f_{r-1}(x)=q_r f_r(x)-f_{r+1}(x),$$

we have
$$f_{r-1}(\alpha)=-f_{r+1}(\alpha),$$

which proves that this value of x gives to $f_{r-1}(x)$ and $f_{r+1}(x)$ the same numerical value with different signs. In passing from a value a little less than α to one a little greater, we can suppose the interval so small that it contains no root of $f_{r-1}(x)$ or $f_{r+1}(x)$; hence, throughout the interval under consideration, these two functions retain their signs. If the sign of $f_r(x)$ does not change (as will happen in the exceptional case when the root α is repeated an even number of times) there is no alteration in the series of signs. In general the sign of $f_r(x)$ changes, but no variation of sign is either lost or gained thereby in the group of three ; because, on account of the difference of signs of the two extremes $f_{r-1}(x)$ and $f_{r+1}(x)$, there will exist both before and after the passage one variation and one permanency of sign, whatever be the sign of the middle function. If, for example, before the passage the signs were $+ \; - \; -$; after the passage they become $+ \; + \; -$, *i.e.*, a variation and a permanency are changed into a permanency and a variation ; but no variation of sign is lost or gained on the whole.

(3) Since the reasoning in the preceding cases is founded on the relations of the function to those adjacent to it only ; and since these relations remain unaltered in the present case, because no two adjacent functions vanish together, we conclude that if $f(x)$ is one of the vanishing functions, one change of sign is lost, and if not, no change is either lost or gained.

We have proved, therefore, that when x passes through a root of $f(x)=0$ one change of sign is lost, and under no other circumstances is a change of sign either lost or gained. Hence the number

of changes of sign lost during the variation of x from a to b is equal to the number of roots of the equation between a and b.*

Before proceeding to the case of equal roots, we add a few simple examples to illustrate the application of Sturm's theorem. It is convenient in practice to substitute first $-\infty$, 0, $+\infty$ in Sturm's functions, so as to obtain the whole number of negative and of positive roots. To separate the negative roots, the integers -1, -2, -3, etc., are to be substituted in succession till we reach the same series of signs as results from the substitution of $-\infty$; and to separate the positive roots we substitute 1, 2, 3, etc., till the signs furnished by $+\infty$ are reached.

Examples

1. Find the number and situation of the real roots of the equation

$$f(x) \equiv x^3-2x-5=0.$$

We find $\quad f_1(x)=3x^2-2, \quad f_2(x)=4x+15, \quad f_3(x)=-643.$

Corresponding to the values $-\infty$, 0, $+\infty$ of x, we have

$$\begin{array}{ll} (-\infty) & - + - -, \\ (0) & - - + -, \\ (+\infty) & + + + -. \end{array}$$

Hence there is only one real root, and it is positive.

Again, corresponding to values of 1, 2, 3 of x, we have

$$\begin{array}{ll} (1) & - + + -. \\ (2) & - + + -. \\ (3) & + + + -. \end{array}$$

The real root, therefore, lies between 2 and 3.

2. Find the number and situation of the real roots of the equation

$$x^3-7x+7=0.$$

We easily obtain

$$\begin{aligned} f_1(x)&=3x^2-7, \\ f_2(x)&=2x-3, \\ f_3(x)&=1\ ; \end{aligned}$$

whence

$$\begin{array}{ll} (-\infty) & - + - +, \\ (0) & + - - +, \\ (+\infty) & + + + +. \end{array}$$

Hence all the roots are real : one negative, and two positive.

*The student often finds a difficulty in perceiving in what way a record is preserved in Sturm's series of the number of changes of sign lost, since the only loss takes place between the first two functions, $f(x)$ and $f_1(x)$. It may tend to remove this difficulty to observe, that as x increases from one root α of $f(x)=0$ to a second β, although no alteration takes place in the *number* of changes of sign, the *distribution* of the signs among $f_1(x)$ and the following functions alters in such a way that the signs of $f(x)$ and $f_1(x)$, which were the same immediately after the passage of x through α, become again different before the passage through β.

We have, further, the following results :—

$$\begin{array}{ll} (-4) & - + - +, \\ (-3) & + + - +, \\ (-2) & + + - +, \\ (-1) & + - - +, \\ (1) & + - - +, \\ (2) & + + + +. \end{array}$$

Here -4 and $+2$ give the same series of signs as $-\infty$ and $+\infty$; hence we stop at these. The negative root lies between -4 and -3; and the two positive roots between 1 and 2.

This example illustrates the superiority of Sturm's method over that of Fourier.

The substitution of 1 and 2 in Fourier's functions gives, as can be immediately verified, the following series of signs :—

$$\begin{array}{ll} (1) & + - + +, \\ (2) & + + + +. \end{array}$$

From Fourier's theorem we are authorised to conclude only that there *cannot be more than* two roots between 1 and 2. From Sturm's we conclude that there *are* two roots between 1 and 2. If we have occasion to separate these two roots, we must, of course, make further substitutions in $f(x)$.

3. Find the number and situation of the real roots of the equation

$$x^4 - 2x^3 - 3x^2 + 10x - 4 = 0.$$

We obtain, removing the factor 2 from the derived,

$$f_1(x) = 2x^3 - 3x^2 - 3x + 5,$$
$$f_2(x) = 9x^2 - 27x + 11.$$
$$f_3(x) = -8x - 3,$$
$$f_4(x) = -1433.$$

[N.B.—In forming Sturm's functions it is allowable, as is evident from the equations (1), to introduce or suppress numerical factors just as in the process of finding the G.C M.; taking care, however, that these are *positive*, so that the signs of the remainders are not thereby altered.]

We have the following series of signs :—

$$\begin{array}{ll} (-\infty) & + - + + -, \\ (0) & - + + - -, \\ (+\infty) & + + + - -. \end{array}$$

Hence there are two real roots, one positive, and one negative, and two imaginary roots. To find the positions of the real roots, it is sufficient to substitute positive and negative integers successively in $f(x)$ alone, since there is only *one* positive and *one* negative root. We easily find in this way that the negative root lies between -2 and -3, and the positive root between 0 and 1.

97. Sturm's Theorem. Equal Roots. Let the operation for finding the greatest common measure of $f(x)$ and $f'(x)$ be performed, the signs of the successive remainders being changed as before. The last of Sturm's functions will not be numerical, for since $f(x)$ and $f'(x)$ are here supposed to contain a common measure involving

x, this will now be the last function arrived at by the process. Let the series of functions be :—

$$f(x), f_1(x), f_2(x), \ldots\ldots, f_r(x).$$

During the passage of x through any value except a multiple root of $f(x)=0$, the conclusions of the last Article are still true with respect to the present series, since no value except such a root can cause any consecutive pair of the series to vanish. When x passes through a multiple root of $f(x)=0$, there is, by the Cor., Art. 75, one change of sign lost between f and f_1 ; and we proceed to prove that no change of sign lost or gained in the rest of the series, *viz.*, $f_1, f_2, \ldots\ldots, f_r$. Suppose there exists an m-multiple root α of $f(x)$. It is evident from the equations (1) of Art. 96, that $(x-\alpha)^{m-1}$ is a factor in each of the functions $f_1, f_2, \ldots\ldots, f_r$. Let the remaining factors in these functions be, respectively, $\phi_1, \phi_2, \ldots\ldots, \phi_r$. By dividing each of the equations (1) by $(x-\alpha)^{m-1}$, we get a series of equations which establish, by the reasoning of the last Article, that, owing to a passage through α, no change of sign is lost or gained in the series $\phi_1, \phi_2, \ldots\ldots, \phi_r$. Neither, therefore, is any change lost or gained in the series $f_1, f_2, \ldots\ldots, f_r$; for the effect of the factor $(x-\alpha)^{m-1}$ in the passage of x from a value $\alpha-h$ to a value $\alpha+h$ is either to change the signs of all (when $m-1$ is odd) or of none (when $m-1$ is even) of the functions $\phi_1, \phi_2, \ldots\ldots, \phi_r$; and changing the signs of all these functions cannot increase or diminish the number of variations.

We have, therefore, proved that when x passes through a multiple root of $f(x)=0$ one change of sign is lost between f and f_1, and none either lost or gained in any other part of the series. It remains true, of course, that when x passes through a single root of $f(x)=0$ a change of sign is lost as before. We may thus state the theorem as follows for the case of equal roots :—

The difference between the number of changes of sign when a and b are substituted in the series

$$f, f_1, f_2, \ldots\ldots, f_r,$$

the last of these being the greatest common measure of f and f_1, is equal to the number of real roots between a and b, each multiple root counting only once.

Examples

1. Find the nature of the roots of the equation

$$x^4-5x^3+9x^2-7x+2=0.$$

We easily obtain

$$f_1(x)=4x^3-15x^2+18x-7,$$
$$f_2(x)=x^2-2x+1\ ;$$

$f_2(x)$ divides $f_1(x)$ without remainder ; hence in this case Sturm's series stops at $f_1(x)$, thus establishing the existence of equal roots.

To find the number of real roots of the equation, we substitute $-\infty$ and $+\infty$ for x in the series of functions f, f_1, f_2. The result is

$$\begin{array}{ll} (-\infty) & + - +, \\ (+\infty) & + + +. \end{array}$$

Hence the equation has only two real distinct roots ; but one of these is a triple root, as is evident from the form of $f_2(x)$, which is equal to $(x-1)^2$.

2. Find the nature of the roots of the equation

$$x^4-6x^3+13x^2-12x+4=0.$$

Here

$$f_1(x)=4x^3-18x^2+26x-12,$$
$$f_2(x)=x^2-3x+2\ ;$$

$f_2(x)$ is the last Sturmian function ; so that the equation has equal roots.

$$\begin{array}{ll} (-\infty) & + - +, \\ (+\infty) & + + +. \end{array}$$

There are only two real distinct roots. In fact, since $f_2(x)\equiv(x-1)(x-2)$, each o the roots 1, 2 is a double root.

3. Find the nature of the roots of the equation

$$x^5+2x^4+x^3-x^2-2x-1=0.$$

Here

$$f_1=5x^4+8x^3+3x^2-2x-2,$$
$$f_2=2x^3+7x^2+12x+7,$$
$$f_3=-x^2-6x-5,$$
$$f_4=-x-1,$$
$$f_5=0.$$

Since $f_5=0$, $x+1$ is a common measure of f and f_1, and $f(x)$ has a double roct -1. We have also

$$\begin{array}{ll} (-\infty) & - + - - +, \\ (+\infty) & + + + - -. \end{array}$$

Hence there are two real distinct roots. The equation has, therefore, beside the double root, one other real root, and two imaginary roots.

4. Find the nature of the roots of the equation

$$x^6-7x^5+15x^4-40x^2+48x-16=0.$$

Here

$$f_1(x)=6x^5-35x^4+60x^3-80x+48,$$
$$f_2(x)=13x^4-84x^3+192x^2-176x+48,$$
$$f_3(x)=x^3-6x^2+12x-8\equiv(x-2)^3.$$

[*Ans.* There are three real distinct roots, one of them being quadruple.

98. Application of Sturm's Theorem. In the case of equations of high degrees the calculation of Sturm's auxiliary functions becomes often very laborious. It is important, therefore, to pay attention to certain observations which tend somewhat to diminish this labour.

(1) In calculating the final remainder when it is numerical, since its sign is all we are concerned with, the labour of the last operation of division can be avoided by the consideration that the value

of x which causes f_{n-1} to vanish must give opposite signs to f_{n-2} and f_n. It is in general possible to tell without any calculation what would be the sign of the result if the root of $f_{n-1}(x)=0$ were substituted in $f_{n-2}(x)$. Thus in Ex. 3, Article 96, if the value $-\frac{3}{8}$, which is the root of $f_3(x)=0$, be substituted for x in $9x^2-27x+11$, the result is evidently positive; hence the sign of $f_n(x)$ is $-$, and there is no occasion to calculate the value -1433 given for $f_n(x)$ in the example referred to.

(2) When it is possible in any way to recognize that all the roots of any one of Sturm's functions are imaginary, we need not proceed to the calculation of any function beyond that one; for since such a function retains constantly the same sign for all values of the variable (Cor. Art. 12), no alteration in the number of changes of sign presented by it and the following functions can ever take place, so that the difference in the number of changes when two quantities a and b are substituted is independent of whatever variations of sign may exist in that part of the series which consists of the function in question and those following it. With a view to the application of this observation it is always well, when we arrive at the quadratic function (ax^2+bx+c, suppose), to examine, in case the term containing x^2 and the absolute term have the same sign (otherwise the roots could not be imaginary), whether the condition $4ac>b^2$ is fulfilled; if so, we know that the roots are imaginary, and the calculation need not proceed farther.

Similar observations apply to the case where one of the functions is a perfect square, since such a function cannot change its sign for real values of x.

Examples

1. Analyse the equation

$$x^4+3x^3+7x^2+10x+1=0.$$

We find

$$f_2(x)=-29x^2-78x+14,$$
$$f_3(x)=-1086x-481,$$
$$f_4(x)=-.$$

Here we see immediately that the value of x given by the equation $f_3(x)=0$, which differs little from $-\frac{1}{2}$, makes $f_2(x)$ positive; hence $f_4(x)$ is negative. There are two real, and two imaginary roots. The real roots lie in the intervals $[-2, -1]$, $[-1, 0]$.

2. Analyse the equation

$$x^4-4x^3-3x+23=0.$$

We find

$$f_2(x)=12x^2+9x-89,$$
$$f_3(x)=-491x+1371,$$
$$f_4(x)=-.$$

Here $f_3(x)=0$ gives $x=\frac{1371}{491}>\frac{1371}{500}>2\cdot74>\frac{5}{2}$, and $x=\frac{5}{2}$ makes $f_2(x)$ positive; hence the root of $f_3(x)$ makes it positive also.

There are two real and two imaginary roots.

The real roots lie in the intervals [2, 3], [3, 4].

3. Analyse the equation

$$2x^4-13x^2+10x+19=0.$$

Here

$$f_1(x)=4x^3-13x+5,$$
$$f_2(x)=13x^2-15x+38.$$

Since $4\times13\times38>15^2$, the roots of $f_2(x)$ are imaginary, and we proceed no further with the calculation of Sturm's remainders.

Substituting $-\infty$, 0, $+\infty$, we obtain

$$\begin{array}{ll}(-\infty) & +\;-\;+,\\ (0) & -\;+\;+,\\ (+\infty) & +\;+\;+.\end{array}$$

There are two real roots, one positive, the other negative.

4. Analyse the equation

$$f(x)\equiv x^5+2x^4+x^3-4x^2-3x-5=0.$$

Here

$$f_1(x)=5x^4+8x^3+3x^2-8x-3,$$
$$f_2(x)=6x^3+66x^2+44x+119,$$
$$f_3(x)=-116x^2-57x-223.$$

Since $4\times116\times223>57^2$, we may stop the calculation here. We find, on substituting $-\infty$, 0, $+\infty$,

$$\begin{array}{ll}(-\infty) & -\;+\;-\;-,\\ (0) & -\;-\;+\;-,\\ (+\infty) & +\;+\;+\;-.\end{array}$$

There are four imaginary roots, and one real positive root.

5 Find the number and situation of the real roots of the equation

$$x^4-2x^3-7x^2+10x+10=0.$$

[*Ans.* The roots are all real, and are situated in the intervals [-3, -2], [-1, 0], and two between [2, 3].

6. Analyse the equation

$$x^5+3x^4+2x^3-3x^2-2x-2=0.$$

It will be found that the calculation may cease with the quadratic remainder.

[*Ans.* There is only one real root; in the interval [1, 2].

7. Analyse the equation

$$x^3+11x^2-102x+181=0.$$

We find

$$f_2(x)=854x-2751,$$
$$f_3(x)=441.$$

In some examples, of which the present is an instance, it is not easy to tell immediately what sign the root of the penultimate function gives to the preceding function. We have here calculated $f_3(x)$, and it turns out to be a much smaller number than might have been expected from the magnitude of the coefficients in $f_2(x)$. In fact when the root of $f_2(x)$ is substituted in $f_1(x)$ the positive part is nearly equal to the negative part. This is always an indication that *two roots of the proposed equation are nearly equal.* There are in the present instance two positive roots between 3 and 4. Subdividing the intervals, we find the two roots still to lie between 3·2 and 3·3; so that they are very close together. We have here another illustration of the continuity which exists between real and imaginary roots (*cf.* Arts. 17, 18). If $f_3(x)$ were zero, the two roots would be equal; and if it were a small negative number, they would be imaginary.

8. Analyse the equation

$$x^5+x^4+x^3-2x^2+2x-1=0.$$

The quadratic function is found to have imaginary roots.

[*Ans.* One real root between [0, 1]; four imaginary roots.

9. Analyse the equation

$$x^6-6x^5-30x^2+12x-9=0.$$

We find

$$f_2(x)=5x^4+20x^3+7;$$

and as this has clearly all imaginary roots, the calculation may stop here.

[*Ans.* Two real roots; in the intervals [−2, −1], [6, 7]

10. Analyse the equation

$$2x^6-18x^5+60x^4-120x^3-30x^2+18x-5=0.$$

We find

$$f_2(x)=5x^4+220x^3+1;$$

and the calculation may stop.

[*Ans.* Two real roots; in the intervals [−1, 0], [5, 6].

11. Examine how the roots of the equation

$$2x^3+15x^2-84x-190=0$$

are situated in the several intervals between the numbers $-\infty$, -7, 6, $+\infty$.

Here

$$f_1(x)=x^2+5x-14,$$
$$f_2(x)=27x+40,$$
$$f_3(x)=+.$$

The substitution of the above quantities gives

$(-\infty)$	$-$	$+$	$-$	$+$,
(-7)	$+$	0	$-$	$+$,
(6)	$+$	$+$	$+$	$+$,
$(-\infty)$	$+$	$+$	$+$	$+$.

Whenever, as in this example, any quantity makes one of the auxiliary functions vanish [here -7 satisfies $f_1(x)=0$], the zero may be disregarded in counting the number of changes of sign in the corresponding row; for, since the signs on each side of it are different, no alteration in the number of changes of signs in the row could take place, whatever sign be supposed attached to the vanishing quantity.

The roots are all real. There is one root between $-\infty$ and -7; and two between -7 and 6.

12. Analyse the equation

$$3x^4-6x^2-8x-3=0.$$

We find

$$f_1(x)=3x^3-3x-2,$$
$$f_2(x)=(x+1)^2.$$

As $f_2(x)$ is a perfect square the calculation may cease.

[*Ans.* Two real roots ; in the intervals $[-1, 0]$, $[1, 2]$.

99. Conditions for the Reality of the Roots of an Equation. The number of Sturm's functions, including $f(x)$, $f'(x)$ and the $n-1$ remainders, will in general be $n+1$. In certain cases, owing to the absence of terms in the proposed functions, some of the remainders will be wanting. This can occur only when the proposed equation has imaginary roots ; for it is clear that, in order to insure a loss of n changes of sign in the series of functions during the passage of x from, $-\infty$ to $+\infty$ (namely, in order that the equation should have all its roots real), all the functions must be present. And, moreover, they must all take the same sign when $x=+\infty$; and alternating signs when $x=-\infty$. Since the leading term of an equation is always taken with a positive sign, we may state the condition for the reality of all the roots of any equation (supposed not to have equal roots) as follows :—*In order that all the roots of an equation of the n^{th} degree should be real, the leading coefficients of all Sturm's remainders, in number $n-1$, must be positive.*

Examples

1. Find the condition that the roots of the equation

$$ax^2+2bx+c=0$$

should be real and unequal. [*Ans.* $b^2-ac>0$.

2. Find the conditions that the roots of the cubic

$$z^3+3Hz+G=0$$

should be all real and unequal.

When this cubic has its roots all real, it is evident that the general cubic from which it is derived (Art. 36) has also its roots all real ; so that in investigating the conditions for the reality of the roots of a cubic in general, it is sufficient to discuss the form here written.

We find

$$f_1(z)=z^2+H,$$
$$f_2(z)=-2Hz-G,$$
$$f_3(z)=-(G^2+4H^3).$$

[In calculating these, before dividing $f_1(z)$ by $f_2(z)$, multiply the former by the positive factor $2H^2$.]

Hence the required conditions are, H negative and G^2+4H^3 negative.

These can be expressed as one condition, *viz.*, G^2+4H^3 negative, since this implies the former (*cf.* Art. 43).

3. Calculate Sturm's remainders for the biquadratic

$$z^4+6Hz^2+4Gz+a^2I-3H^2=0.$$

We find
$$f_2(z) = -3Hz^2 - 3Gz - (a^2I - 3H^2),$$
$$f_3(z) = -(2HI - 3aJ)z - GI,$$
$$f_4(z) = I^3 - 27J^2.$$

These are obtained without much difficulty by aid of the identity of Art 37. Before dividing f_1 by f_2, multiply by the positive factor $3H^2$; and when the remainder is found, remove the positive factor a^2. Before dividing f_2 by f_3, multiply by the positive factor $(2HI - 3aJ)^2$; and when the remainder is found, remove the positive factor a^2H^2.

100. Conditions for the Reality of the Roots of a Biquadratic. In order to arrive at criteria of the nature of the roots of the general algebraic equation of the fourth degree by Sturm's method, it is sufficient to consider the equation of Ex. 3 of the preceding Article. By aid of the forms of the leading co-efficients of Sturm's remainders there calculated, we can write down the *conditions that all the roots of a biquadratic should be real and unequal* in the form

H negative, $2HI - 3aJ$ negative, $I^3 - 27J^2$ positive.

It will be observed that the second of these conditions is different in form from the corresponding condition of Art. 68. To show the equivalence of the two forms it is necessary to prove that when H is negative and Δ positive, the further condition $2HI - 3aJ$ negative implies the condition $a^2I - 12H^2$ negative, and conversely. From the identity of Art. 37, written in the form $-H(a^2I - 12H^2) \equiv a^2(2HI - 3aJ) - 3G^2$, it readily appears that when H and $2HI - 3aJ$ are negative $a^2I - 12H^2$ is necessarily negative. And to prove the converse we observe that when aJ is positive $2HI - 3aJ$ is negative, since I is positive on account of the condition Δ positive; and when aJ is negative $2HI - 3aJ$ is still negative, since the negative part $2HI$ exceeds the positive part $-3aJ$, as may be readily shown by the aid of the inequalities $12H^2 > a^2I$ and $I^3 > 27J^2$.

The student will have no difficulty in verifying, by means of Sturm's functions, the remaining conclusions arrived at in the different cases of Art. 68.

Examples

1. Apply Budan's method to separate the roots of the equation
$$x^4 - 16x^3 + 69x^2 - 70x - 42 = 0.$$
[*Ans.* Roots in intervals $[-1, 0]$, $[2, 3]$, $[4, 5]$, $[9, 10]$.

2. Apply Sturm's theorem to the analysis of the equation
$$x^4 - 4x^3 + 7x^2 - 6x - 4 = 0.$$

In analysing a biquadratic of this nature which has clearly two real roots, when a Sturmian remainder is reached whose leading co-efficient is negative, the calculation may cease, since the other pair of roots must then be imaginary, and the positions of the real roots may be readily found by substitution in the given equation. [*Ans.* Two roots imaginary; real roots in intervals $[-1, 0]$, $[2, 3]$.

3. Analyse in a similar manner the equation

$$x^4-5x^3+10x^2-6x-21=0.$$

[*Ans.* Two roots imaginary ; real roots in intervals [−1, 0] ; [3, 4].

4. Apply Sturm's theorem to the analysis of the equation

$$x^4+3x^3-x^2-3x+11=0.$$

[*Ans.* Roots all imaginary.

5. Find, by Sturm's method, the number and positions of the real roots of the equation

$$x^5-10x^3+6x+1=0.$$

[*Ans.* Roots all real ; one in the interval [−4, −3] ; two in the interval [−1, 0] ; and positive roots in the intervals [0, 1], [3 [illegible].

6 Calculate Sturm's functions for the following equation, and show that all the roots are real :—

$$x^5-5x^4+5x^3+5x^2-5x-1=0.$$

7. Calculate Sturm's functions for the following equation, and show that four roots are imaginary :—

$$3x^5+5x^3+2=0.$$

This and the preceding example are instances in which, as the student will easily see, there is a factor common to two of Sturm's remainders which are not consecutive.

8. Calculate Sturm's functions for the following equation, and verify the conclusions of Ex. 23, p. 85, with regard to the character of the roots :—

$$x^5-5px^3+5p^2x+2q=0.$$

9. Prove that, if c has any value except unity, the equation

$$c^2x^4-2c^2x^3+2x-1=0$$

has a pair of imaginary roots

10. Prove that the roots of the equation

$$x^3-(a^2+b^2+c^2)x-2abc=0$$

are all real, and solve it when two of the quantities a, b, c become equal.

11. Prove that when the biquadratic

$$f(x)\equiv ax^4+4bx^3+6cx^2+4dx+e$$

has a triple factor, it may be expressed in the form

$$a^3f(x)\equiv[ax+b+\sqrt{-H}]^3[ax+b-3\sqrt{-H}].$$

12. Verify, by means of Sturm's remainders, the conditions which must be fulfilled when the biquadratic of the preceding example is a perfect square, and prove in that case

$$a^3f(x)\equiv[(ax+b)^2+3H]^2.$$

13. Prove that, when all Sturm's functions are present, the number of changes of sign among the coefficients of the leading terms is equal to the number of pairs of imaginary roots of the equation.

14. If the signs of the leading coefficients of the first two of Sturm's remainders for a quintic be − +, prove that the number of real roots is determined. [*Ans.* One real root only.

15. If H and J are both positive, prove that all the roots of the biquadratic are imaginary ; and that under the same conditions the qui[illegible] [illegible]en w[illegible]h binomial coefficients has only one real root.

(Mr M. Roberts, *Dublin Exam. Papers*, 1862)

16. In the application of Sturm's theorem, if any function be reached whose signs are all positive or all negative, the number and situations of the *positive* roots of the original equation can be examined without the aid of the lower

Sturmians; and if a function be reached whose signs are alternately positive and negative, the *negative* roots of the original equation may be discussed in a similar manner.

17. If all the roots of any equation $f(x)=0$ are real, prove that all the roots of every one of Sturm's auxiliary functions are also real.

This can be established by reasoning similar to that of Art. 96. Consider the k^{th} remainder R_k, and let its degree be m. This and the m functions which follow constitute a series of which no adjacent two can vanish together. When $x=-\infty$, their signs are alternately positive and negative, and when $x=+\infty$, they are all positive. There are, therefore, m changes of sign to be lost as x passes from $-\infty$ to $+\infty$; and no change of sign canbe lost except on the passage through a root of $R_k=0$, which equation must consequently have m real roots.

Since a value of x which causes any of the functions to vanish gives opposite signs to the two adjacent functions, it is easily inferred that any equation of the series is a limiting equation with regard to the function which precedes it.

18. If the real roots of any one, $f_m(x)$, of the Sturmian auxiliary functions be known, prove that the number and positions of the roots of the original equation may be determined without the aid of the functions below $f_m(x)$.

Let the real roots, in order of magnitude, of $f_m(x)=0$ be α, β,........, η, θ; the remaining roots being imaginary. As x varies from $-\infty$ to a value a little less than θ, the function $f_m(x)$ cannot change its sign; and, therefore, in examining the roots of $f(x)=0$ which lie between these limits, the Sturmians which follow $f_m(x)$ may be disregarded. The same holds true as x passes from a value a little greater than θ to one a little less than η; and similarly for the remaining intervals. If, therefore, we examine separately the intervals $[-\infty, \theta]$, $[\theta, \eta]$...... $[\beta, \alpha]$, $[\alpha, +\infty]$, the number of roots of the original equation which lie in each of these regions can be determined without the aid of the lower Sturmian functions.

19. If any one of Sturm's auxiliary functions has imaginary roots, the original equation has at least an equal number of imaginary roots.

(Mr. F. PURSER)

This can be inferred from the preceding example by examining the greatest possible number of changes which can be lost in the series terminating with $f_m(x)$, during the passage of x from $-\infty$ to $+\infty$; remembering that, so far as the limited series is concerned, a change of sign may be gained on the passage through each real root of $f_m(x)=0$.

20. Apply the method of Ex. 18 to the equation of Ex. 1, Art. 98.

Disregarding the two lowest Sturmian remainders, we have

$$f(x)\equiv x^4+3x^3+7x^2+10x+1,$$
$$f'(x)\equiv 4x^3+9x^2+14x+10,$$
$$R_1\equiv -29x^2-78x+14.$$

The roots of the equation $R_1=0$ are easily seen to lie in the intervals $(-3, -2)$ and $(0, 1)$. The equation $f(x)=0$ has two imaginary roots, since the coefficient of x^2 in R_1 is negative. The real roots, if any, must be negative. The three functions above written are sufficient to determine the existence and situations of roots in the intervals $(-\infty, -3)$ and $(-2, 0)$. It is at once seen the two real roots of the original equation are situated in the latter interval.

It will be found possible in many examples to avoid in this way the calculation of the last two Sturmian remainders; and it will be observed that it is not necessary to know the actual roots of the quadratic function, but only the intervals in which they are situated.

CHAPTER XI

SOLUTION OF NUMERICAL EQUATIONS

101. Algebraical and Numerical Equations. There is an essential distinction between the solutions of algebraical and numerical equations. In the former the result is a general formula of a purely symbolical character, which, being the general expression for a root, must represent all the roots indifferently. It must be such that, when for the functions of the co-efficients involved in it the corresponding symmetric functions of the roots are substituted, the operations represented by the radical signs $\sqrt{\ }$, $\sqrt[3]{\ }$ become practicable; and when the square and cube roots of these symmetric functions are extracted, the whole expression in terms of the roots will reduce down to one root: the different roots resulting from the different combinations $\pm\sqrt{\ }$ of square roots, and $\sqrt[3]{\ }$, $\omega\sqrt[3]{\ }$, $\omega^2\sqrt[3]{\ }$ of cube roots. For a simple illustration of what is here stated, we refer to the case of the quadratic in Art. 55. In Articles 59 and 66 we have similar illustrations for the cubic and biquadratic. It is to be observed also that for formula which represents the root of an algebraic equation holds good even when the co-efficients are imaginary quantities.

In the case of numerical equations the roots are determined separately by the methods we are about to explain; and, before attempting the approximation to any individual root, it is in general necessary that it should be situated in a known interval which contains no other real root.

The real roots of numerical equations may be either commensurable or incommensurable; the former class including integers, fractions, and terminating or repeating decimals, which are reducible to fractions; the latter consisting of interminable decimals. The roots of the former class can be found exactly, and those of the latter approximated to with any degree of accuracy, by the methods we are about to explain.

We shall commence by establishing a theorem which reduces the determination of the former class of roots to that of *integer roots* alone.

102. Theorem. *An equation in which the co-efficient of the first term is unity, and the co-efficients of the other terms whole numbers, cannot have a commensurable root which is not a whole number.*

For, if possible, let $\frac{a}{b}$, a fraction in its lowest terms, be a root of the equation

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots\ldots + p_{n-1}x + p_n = 0\ ;$$

we have then

$$\left(\frac{a}{b}\right)^n + p_1\left(\frac{a}{b}\right)^{n-1} + \ldots\ldots + p_{n-1}\left(\frac{a}{b}\right) + p_n = 0\ ;$$

from which, multiplying by b^{n-1}, we obtain

$$-\frac{a^n}{b} = p_1a^{n-1} + p_2a^{n-2}b + \ldots\ldots + p_{n-1}ab^{n-2} + p_nb^{n-1}$$

Now a^n is not divisible by b, and each term on the right-hand side of the equation is an integer. We have, therefore, a fraction in its lowest terms equal to an integer, which is impossible. Hence $\frac{a}{b}$ cannot be a root of the equation. The real roots of the equation, therefore, are either integers or incommensurable quantities.

Every equation whose co-efficients are finite numbers, fractional or not, can be reduced to the form in which the co-efficient of the first term is unity and those of the other terms whole numbers (Art. 31) ; so that in this way, by the aid of a simple transformation, the determination of the commensurable roots in general can be reduced to that of integer roots.

We proceed to explain Newton's process, called the Method of Divisors, of obtaining the integer roots of an equation whose co-efficients are all integers.

103. Newton's Method of Divisors. Suppose h to be an integer root of the equation

$$a_0x^n + a_1x^{n-1} + \ldots\ldots + a_{n-1}x + a_n = 0. \qquad \ldots(1)$$

Let the quotient, when the polynomial is divided by $x-h$, be

$$b_0x^{n-1} + b_1x^{n-2} + \ldots\ldots + b_{n-2}x + b_{n-1},$$

in which b_0, b_1, etc., are clearly all integers.

Proceeding as in Art. 8, we obtain the following equations :—

$$a_0 = b_0,\ a_1 = b_1 - hb_0,\ a_2 = b_2 - hb_1, \ldots\ldots$$

$$a_{n-2} = b_{n-2} - hb_{n-3},\ a_{n-1} = b_{n-1} - hb_{n-2},\ a_n = -hb_{n-1}.$$

The last of these equations proves that a_n is divisible by h, the quotient being $-b_{n-1}$. The second last, which is the same as

$$a_{n-1}+\frac{a_n}{h}=-hb_{n-2},$$

proves that the sum of the quotient thus obtained and the second last co-efficient is again divisible by h, the quotient being $-b_{n-2}$; and so on.

Continuing the process, the last quotient obtained in this way will be $-b_0$, which is equal to $-a_0$.

If we perform the process here indicated with all the divisors of a_n which lie within the limits of the roots, those which satisfy the above conditions, giving integer quotients at each step, and a final quotient equal to $-a_0$, are roots of the proposed equation. Those which at any stage of the process give a fractional quotient are to be rejected.

When the co-efficient $a_0=1$, we know by the theorem of the last Article that the integer roots determined in this way are all the commensurable roots of the proposed equation. If a_0 be not $=1$, the process will still give the integer roots of the equation as it stands; but to be sure of determining in this way all the commensurable roots, we must first transform the equation to one which shall have the co-efficient of the highest term equal to unity.

104. Application of the Method of Divisors. With a view to the most convenient mode of applying the Method of Divisors, we write the series of operations as follows, in a manner analogous to Art. 8 :—

$$\begin{array}{ccccc} a_{n-1} & a_{n-2} \cdot & \cdot a_2 & a_1 & \\ -b_{n-1} & -b_{n-2} & -b_2 & -b_1 & -b_0 \\ \hline -hb_{n-2} & -hb_{n-3} & -hb_1 & -hb_0 & 0 \end{array}$$

The first figure in the second line $(-b_{n-1})$ is obtained by dividing a_n by h. This is to be added to a_{n-1} to obtain the first figure in the third line $(-hb_{n-2})$. This is to be divided by h to obtain the second figure in the second line $(-b_{n-2})$; this is to be added to a_{n-2}, and so on. If h be a root, the last figure in the second line thus obtained will be $-a_0$.

When we succeed in proving in this manner that any integer h is a root, the next operation with any divisor may be performed, not on the original co-efficients $a_n, a_{n-1}, \ldots\ldots$, but on those of the second line with their signs changed, for these are the co-efficients of the quotient when the original polynomial is divided by $x-h$. When any

divisor gives at any stage a fractional result it is to be rejected at once, and the operation so far as it is concerned stopped.

The numbers 1 and -1, which are always of course integer divisors of a_n, need not be included in the number of trial divisors. It is more convenient before applying the Method of Divisors to determine by direct substitution whether either of these numbers is a root.

Examples

1. Find the integer roots of the equation

$$x^4-2x^3-13x^2+38x-24=0.$$

By grouping the terms (*see* Art. 87) we observe without difficulty that all the roots lie between -5 and $+5$. The following divisors are possible roots :—

$$-4, \quad -3, \quad -2, \quad 2, \quad 3, \quad 4.$$

We commence with 4 :—

$$\begin{array}{ccccc} -24 & 38 & -13 & -2 & 1 \\ & -6 & 8 & & \\ \hline & 32 & -5 & & \end{array}$$

The operation stops here, for since -5 is not divisible by 4, 4 cannot be a root.

We proceed then with the number 3 :

$$\begin{array}{ccccc} -24 & 38 & -13 & -2 & 1 \\ & -8 & 10 & -1 & -1 \\ \hline & 30 & -3 & -3 & 0\,; \end{array}$$

hence 3 is a root ; and in proceeding with the next integer, 2, we make use, as above explained, of the co-efficients of the second line with signs changed :

$$\begin{array}{cccc} 8 & 10 & 1 & 1 \\ & 4 & -3 & -1 \\ \hline & -6 & -2 & 0\,; \end{array}$$

hence 2 also is a root ; and we proceed with -2 :

$$\begin{array}{ccc} -4 & 3 & 1 \\ & 2 & \\ \hline & 5\,; & \end{array}$$

hence -2 is not a root, for it does not divide 5. -3 is plainly not a root, for it does not divide -4.

[We might at once have struck out -3 as not being a divisor of the absolute term 8 of the reduced polynomial. This remark will often be of use in diminishing the number of divisors.]

We proceed now to the last divisor, -4.

$$\begin{array}{ccc} -4 & 3 & 1 \\ & 1 & -1 \end{array}$$

Thus -4 is a root.

The equation has, therefore, the integer roots 3, 2, -4; and the last stage of the operation shows that when the original polynomial is divided by the binomials $x-3$, $x-2$, $x+4$, the result is $x-1$; so that 1 is also a root. Hence the original polynomial is equivalent to

$$(x-1)(x-2)(x-3)(x+4).$$

2. Find the integer roots of

$$3x^4-23x^3+35x^2+31x-30=0.$$

The roots lie between -2 and 8; hence we have only to test the divisors 2, 3, 5, 6.

We find immediately that 6 is not a root.

For 5 we have

$$\begin{array}{rrrrr} -30 & 31 & 35 & -23 & 3 \\ & -6 & 5 & 8 & -3 \\ \hline & 25 & 40 & -15 & 0 \end{array}$$

hence 5 is a root. For 3 we have

$$\begin{array}{rrrr} 6 & -5 & -8 & 3 \\ & 2 & -1 & -3 \\ \hline & -3 & -9 & 0, \end{array}$$

hence 3 is a root; and we easily find that 2 is not a root.

The quotient, when the original polynomial is divided by $(x-5)(x-3)$, is, from the last operation,

$$3x^2+x-2;$$

of this 1 is not a root, and -1 is; so that all the integer roots of the proposed equation are -1, 3, 5.

The other root of the equation is $2/3$; it is a commensurable root; but, not being an integer, is not given in the above operation.

3. Find all the roots of the equation

$$x^4+x^3-2x^2+4x-24=0.$$

Limits of the roots are -4, 3. [*Ans.* Roots -3, 2, $\pm 2\sqrt{-1}$.

4. Find all the roots of the equation

$$x^4-2x^3-19x^2+68x-60=0.$$

The roots lie between -6 and 6.

We find that 2, 3, -5 are roots, and that the factor left after the final division is $x-2$; hence 2 is a double root. The polynomial is, therefore, equivalent to

$$(x-2)^2(x+3)(x+5).$$

In Art. 106 the case of multiple roots will be further considered.

105. Method of Limiting the Number of Divisors. It is possible of course to determine by direct substitution whether any of the divisors of a_n are roots of the proposed equation; but Newton's method has the advantage, as the above examples show, that some

of the divisors are rejected after very little labour. It has a further advantage which will now be explained. When the number of divisors of a_n within the limits of the roots is large, it is important to be able, before proceeding with the application of the method in detail, to diminish the number of these divisors which need be tested. This can be done as follows :—

If h is an integer root of $f(x)=0$, $f(x)$ is divisible by $x-h$ and the coefficients of the quotient are integers, as was above explained. If, therefore, we assign to x any integer value, the quotient of the corresponding value of $f(x)$ by the corresponding value of $x-h$ must be an integer. We take for convenience, the simplest integers 1 and -1; and before testing any divisor h, we subject it to the condition that $f(1)$ must be divisible by $1-h$ (or changing the sign, by $h-1$); and that $f(-1)$ must be divisible by $-1-h$ (or, changing the sign, by $1+h$).

In applying this observation it will be found convenient to calculate $f(1)$ and $f(-1)$ in the first instance if either of these vanishes, the corresponding integer is a root, and we proceed with the operation on the reduced polynomial whose coefficients have been ascertained in the process of finding the result of substituting the integer in question.

Examples

1. $$x^5-23x^4+160x^3-281x^2-257x-440=0.$$

The roots lie between -1 and 24.

We have the following divisors :—

2, 4, 5, 8, 10, 11, 20, 22.

We easily find

$$f(1)=-840, \text{ and } f(-1)=-648.$$

We, therefore, exclude all the above divisors, which, when diminished by 1, do not divide 840 ; and which when increased by 1, do not divide 648. The first condition excludes 10 and 20, and the second 4 and 22. Applying the Method of Divisors to the remaining integers 2, 5, 8, 11, we find that 5, 8 and 11 are roots and that the resulting quotient is x^2+x+1. Hence the given polynomial is equivalent to

$$(x-5)(x-8)(x-11)(x^2+x+1).$$

2. $$x^5-29x^4-31x^3+31x^2-32x+60=0.$$

The roots lie between -3 and 32.

Divisors : -2, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30.

$f(1)=0$; so 1 is a root.

$f(-1)=124$; and the above condition excludes all the divisors except -2, 3, 30.

We easily find that -2 and 30 are roots and that the final quotient is x^2+1. The given polynomial is equivalent to

$$(x-1)(x-30)(x+2)(x^2+1).$$

106. Determination of Multiple Roots. The Method of Divisors determines multiple roots when they are commensurable. In applying the method, when any divisor of a_n which is found to be a root is a divisor of the absolute term of the reduced polynomial, we must proceed to try whether it is also a root of the latter, in which case it will be a double root of the proposed equation. If it be found to be a root of the next reduced polynomial, it will be a triple root of the proposed ; and so on. Whenever in an equation of any degree there exists only *one* multiple root, r times repeated, it can be found in this way ; for the common measure of $f(x)$ and $f'(x)$ will then be of the form $(x-\alpha)^{r-1}$, and the coefficients of this could not be commensurable if α were incommensurable.

Multiple roots of equations of the third, fourth, and fifth degrees can be completely determined without the use of the process of finding the greatest common measure, as will appear from the following observations :—

(1) *The Cubic.* In this case multiple roots must be commensurable, since the degree is not high enough to allow of two distinct roots being repeated.

(2) *The Biquadratic.* In this case either the multiple roots are commensurable or the function is a perfect square. For the only form of biquadratic which admits of two distinct roots being repeated is

$$(x-\alpha)^2(x-\beta)^2,$$

viz., the square of a quadratic. The roots of the quadratic may be incommensurable. If we find, therefore, that a biquadratic has no commensurable roots, we must try whether it is a perfect square in order to determine further whether it has equal incommensurable roots.

(3) *The Quintic.* In this case, either the multiple roots are commensurable, or the function consists of a linear commensurable factor multiplied by the square of a quadratic factor. For, in order that two distinct roots may be repeated, the function must take one or other of the forms

$$(x-\alpha)^2(x-\beta)^2(x-\gamma),\ (x-\alpha)^3(x-\beta)^2.$$

In the latter case the roots cannot be incommensurable ; but the former may correspond to the case of a commensurable factor multiplied by the square of a quadratic whose roots are incommensurable. If then a quintic be found to have no commensurable roots, it can have no multiple roots. If it be found to have one commensurable root only, we must examine whether the remaining factor is a perfect

square. *If it have more than one commensurable root, the multiple roots will be found among the commensurable roots.*

Examples

1. Find all the commensurable roots of

$$2x^3-31x^2+112x+64=0.$$

The roots lie between the limits -1, 16. The divisors are 2, 4, 8.

64	112	-31	2
	8	15	-2
	120	-16	0 ;

8 is, therefore, a root. Proceed now with the reduced equation :

-8	-16	2
	-1	-2
	-16	0 ,

8 is a root again, and the remaining factor is $2x+1$.

[*Ans.* $f(x)\equiv(2x+1)(x-8)^2$.

2. Find the commensurable and multiple roots of

$$x^4-x^3-30x^2-76x-56=0.$$

The roots lie between the limits -6, 12. (Apply method of Ex. 10, p. 87.) [*Ans.* $f(x)\equiv(x+2)^3(x-7)$.

3. Find the commensurable and multiple roots of

$$9x^4-12x^3-71x^2-40x+16=0.$$

The roots lie between the limits -2, 5.

The equation as it stands is found to have no integer root ; but it may still have a commensurable root. To test this we multiply the roots by 3 in order to get rid of the coefficient of x^4. We find then

$$x^4-4x^3-71x^2-120x+144=0.$$

Limits -6, 15.

We find -4 to be a double root of this, and the function to be equivalent to $(x^2-12x+9)(x+4)^2$. The original equation is, therefore, identical with the following : –

$$(x^2-4x+1)(3x+4)^2=0.$$

4. Find the commensurable and multiple roots of

$$x^4+12x^3+32x^2-24x+4=0.$$

The roots lie between -12 and 1. The only divisors to be tested are, therefore, -4, -2, -1. We find that the equation has no commensurable root. We proceed to try whether the given function is a perfect square. This can be done by extracting the square root, or by applying the conditions of Ex. 3, p. 102. We find that it is the square of x^2+6x-2 (*cf.* Ex. 1, p. 129). Hence the given equation has two pairs of equal roots, both incommensurable.

5. Find the commensurable and multiple roots of

$$f(x)\equiv x^5-x^4-12x^3+8x^2+28x+12=0.$$

The limits of the roots are -4, 4.

We find that -3 is a root, and that the reduced equation is

$$x^4-4x^3+8x+4=0,$$

and that there is no other commensurable root.

The only case of possible occurrence of multiple roots is, therefore, when this latter function is a perfect square. It is found to be perfect square, and we have

$$f(x) \equiv (x^2-2x-2)^2(x+3)^2.$$

6. Find the commensurable and multiple roots of

$$f(x) \equiv x^5-8x^4+22x^3-26x^2+21x-18=0.$$

[*Ans.* $f(x) \equiv (x^2+1)(x-2)(x-3)^2$.

7. The following equation has only two different roots : find them :—

$$x^5-13x^4+67x^3-171x^2+216x-108=0.$$

In general it is obvious that if an integer root h occurs twice, the last coefficient must contain h^2 as a factor, and the second last h ; if the root occurs three times, h^3 must be a factor of the last, h^2 of the second last and h of the third last coefficient. The last coefficient here $=2^2.3^3$. Hence, if neither -1 nor 1 is a root, the required roots must be 2 and 3. That these are the roots is easily verified.

8. The equation

$$800x^4-102x^2-x+3=0$$

has equal roots ; find all the roots.

In this example it is convenient to change the roots into their reciprocals before applying the Method of Divisors.

[*Ans.* $f(x) \equiv (10x-3)(5x-1)(4x+1)^2$.

107. Newton's Method of Approximation. Having shown how the commensurable roots of equations may be obtained, we proceed to give an account of certain methods of obtaining approximate values of the incommensurable roots. The method of approximation, commonly ascribed to Newton,* which forms the subject of the present Article, is valuable as being applicable to numerical equations involving transcendental functions, as well as those which involve algebraical functions only. Although when applied to the latter class of functions Newton's method is, for practical purposes, inferior in form to Horner's, which will be explained in the following Articles, yet in principle both methods are to a great extent identical.

In all methods of approximation the root we are seeking is supposed to be separated from the other roots, and to be situated in a known interval between close limits.

Let $f(x)=0$ be a given equation, and suppose a value a to be known, differing by a small quantity h from a root of the equation. We have then, since $a+h$ is a root of the equation, $f(a+h)=0$; or

$$f(a)+f'(a)h+\frac{f''(a)}{1.2}h^2+\ldots\ldots=0.$$

Neglecting now, since h is small, all powers of h higher than the first, we have

$$f(a)+f'(a)\,h=0,$$

*See Note B at the end of the volume.

giving, as a first approximation to the root, the value

$$a-\frac{f(a)}{f'(a)}.$$

Representing this value by b, and applying the same process second time, we find as a closer approximation

$$b-\frac{f(b)}{f'(b)}.$$

By repeating this process the approximation can be carried to any degree ef accuracy required.

Example

Find on approximate value of the positive root of the equation

$$x^3-2x-5=0.$$

The root lies between 2 and 3 (Ex. 1, Art. 96). Narrowing the limits, the root is found to lie between 2 and 2·2. We take 2·1 as the quantity represented by a. It cannot differ from the true value $a+h$ of the root by more than 0·1. We find easily

$$\frac{f(a)}{f'(a)}=\frac{f(2\cdot1)}{f'(2\cdot1)}=\frac{\cdot061}{11\cdot23}=0\cdot00543.$$

A first approximation is, therefore,

$$2\cdot1-0\cdot00543=2\cdot0946.$$

Taking this as b, and calculating the fraction $\frac{f(b)}{f'(b)}$, we obtain

$$b-\frac{f(b)}{f'(b)}=2\cdot09455148,$$

for a second approximation ; and so on.

The approximation in Newton's method is, in general, rapid. When, however, the root we are seeking is accompanied by another nearly equal to it, the fraction $\frac{f(a)}{f'(a)}$ is not necessarily small, since the value of either of the nearly equal roots reduces $f'(x)$ to a small quantity. A case of this kind requires special precautions. We do not enter into any further discussion of the method, since for practical purposes it may be regarded as entirely superseded by Horner's method, which will now be explained.

108. Horner's Method of Solving Numerical Equations. By this method both the commensurable and incommensurable roots can be obtained. The root is evolved figure by figure : first the integer part (if any), and then the decimal part, till the root terminates if it be commensurable, or to any number of places required if it be incommensurable. The process is similar to the known processes of extraction of the square and cube root, which are, indeed, only particular cases of the general solution by the present method of quadratic and cubic equations.

The main principle involved in Horner's method is the successive diminution of the roots of the given equation by known quantities, in the manner explained in Art. 33. The great advantage of the method is, that the successive transformations are exhibited in a compact arithmetical form, and the root obtained by one continuous process correct to any number of places of decimals required.

This principle of the diminution of the roots will be illustrated in the present Article by simple examples. In the Articles which follow, some additional principles which tend to facilitate the practical application of the method will be explained.

Examples

1. Find the positive root of the equation

$$2x^3-85x^2-85x-87=0.$$

The first step, when any numerical equation is proposed for solution, is to find the *first figure* of the root. This can usually be done by a few trials; although in certain cases the methods of separation of the roots explained in Chap. X, may have to be employed. In the present example there can be only one positive root; and it is found by trial to lie between 40 and 50. Thus the first figure of the root is 4. We now diminish the roots by 40. The transformed equation will have one root between 0 and 10. It is found by trial to lie between 3 and 4. We now diminish the roots of the transformed equation by 3; so that the roots of the proposed equation will be diminished by 43. The second transformed equation will have one root between 0 and 1. On diminishing the roots of this latter equation by ·5, we find that its absolute term is reduced to zero, *i.e.*, the diminution of the roots of the proposed equation by 43·5 reduces its absolute term to zero. We conclude that 43·5 is a root of the given equation. The series of arithmetical operations is represented as follows :—

2	−85	−85	−87	(43·5
	80	−200	−11400	
	−5	−285	**−11487**	
	80	3000	9594	
	75	**2715**	**−1893**	
	80	483	1893	
	155	**3198**		
	6	501		
	161	**3699**		
	6	87		
	167	3786		
	6			
	173			
	1			
	174			

The broken lines mark the conclusion of each transformation, and the figures in dark type are the coefficients of the successive transformed equations (*see* Art. 33). Thus

$$2x^3+155x^2+2715x-11487=0,$$

is the equation whose roots are each less by 40 than the roots of the given equation, and whose positive root is found to lie between 3 and 4. If the second transformed equation had not an exact root ·5; but one, we shall suppose, between ·5 and ·6, the first three figures of the root of the proposed equation would be 43·5 and to find the next figure we should proceed to a further transformation, diminishing the roots by ·5; and so on.

2. Find the positive root of the equation

$$4x^3-13x^2-31x-275=0.$$

We first write down the arithmetical work, and proceed to make certain observations on it:—

4	−13	−31	−275	(6·25
	24	66	210	
	11	35	**−65**	
	24	210	51·392	
	35	**245**	**−13·608**	
	24	11·96	13·608	
	59	256·96	0	
	·8	12·12		
	59·8	**269·08**		
	·8	3·08		
	60·6	272·16		
	·8			
	61·4			
	·2			
	61·6			

We find by trial that the proposed equation has its positive root between 6 and 7. The first figure of the root is, therefore, 6. Diminish the roots by 6. The equation

$$x^3+59x^2+245x-65=0$$

has, therefore, a root between 0 and 1. It is found by trial to lie between ·2 and ·3. The first two figures of the root of the proposed equation are, therefore, 6·2. Diminish the roots again by ·2. The transformed equation is found to have the root ·05. Hence 6·25 is a root of the proposed equation.

It is convenient in practice to avoid the use of the decimal points. This can easily be effected as follows:—When the decimal part of the root (suppose ·*abc*...) is about to appear, multiply the roots of the corresponding transformed equation by 10, *i.e.*, annex one zero to the right of the figure in the first column, two to the right of the figure in the second column, three to the right of that in the third; and so on, if there be more columns (as there will of course be in equations of a degree higher than the third). The root of the transformed equation is then, not ·*abc*......, but *a*·*bc*... Diminish the roots by *a*. The transformed equation has a root ·*bc*... Multiply the roots of this equation again by 10, The root becomes *b*·*c*..., and the process is continued as before. To illustrate this

we repeat the above operation, omitting the decimal points. In all subsequent examples this simplification will be adopted :—

-13	-31	-275	(6·25
24	66	210	
11	35	-65000	
24	210	51392	
35	**24500**	$-$**13608000**	
24	1196	13608000	
590	25696	0	
8	1212		
598	**2690800**		
8	30800		
606	2721600		
8			
6140			
20			
6160			

3. Find the positive root of the equation

$$20x^3-121x^2-121x-141=0.$$

The root is easily found to lie between 7 and 8. It is, therefore, of the form $7.ab\ldots$ When the roots are diminished by 7, and multiplied by 10, the resulting equation is

$$20x^3+2990x^2+112500x-57000=0.$$

The positive root of this is $a{\cdot}b$ .. ; and as the root clearly lies between 0 and 1, we have $a=0$. We, therefore, place zero as the first figure in the decimal part of the root, and multiply the roots again by 10, before proceeding to the second transformation. 5 is easily seen to be a root of the equation thus transformed.

[*Ans.* 7·05.

In the examples here considered the root terminates at an early stage. When the calculation is of greater length, if it were necessary to find the successive figures by substitution, the labour of the process would be very great. This, however, is not necessary, as will appear in the next article ; and one of the most valuable practical advantages of Horner's method is, that after the second, or third (sometimes even after the first) figure of the root is found, the *transformed equation itself suggests by mere inspection the next figure of the root.* The principle of this simplification will now be explained.

109. Principle of the Trial divisor. We have seen in Art. 107 that when an equation is transformed by the substitution of $a+h$ for, x, a being a number differing from the true root by a quantity h small in proportion to a, an approximate numerical value of h is obtained by dividing $f(a)$ by $f'(a)$. Now the successive transformed equations in Horner's process are the results of transformations of this kind, the last coefficient being $f(a)$, and the second last $f'(a)$ (*see* Art. 33). Hence, after two or three steps have been completed, so that the part of the root remaining bears a small ratio to the part already evolved, we may expect to be furnished with two or three more figures of the root correctly by mere division of the last by the second last coefficient of the final transformed equation. We might, therefore, if we pleased, at any stage of Horner's operations, apply Newton's method to get a further approximation to the root. In Horner's method this principle is employed to suggest the next follow-

ing figure of the root after the figures already obtained. The second last coefficient of each transformed equation is called the *trial-divisor.* Thus, in the second example of the last Article, the number 5 is correctly suggested by the trial-divisor 2690800. In this example, indeed, the second figure of the root is correctly suggested by the trial-divisor of the first transformed equation ; although, in general, such is not the case. In practice the student will have to estimate the probable effect of the leading coefficients of the transformed equation ; he will find, however, that the influence of these terms becomes less and less as the evolution of the root proceeds.

Examples

1. Find the positive root of the equation

$$x^3+x^2+x-100=0$$

correct to four decimal places.

It is easily seen that the root lies between 4 and 5. We write down the work, and proceed to make observations on it :—

1	1	1	−100	(4·2644
	4	20	84	
		21	**−16000**	
		36	11928	
		5700	**−4072000**	
		264	3788376	
	130	5964	**−283624000**	
	2	268	256071744	
	132	**623200**	**−27552256**	
	2	8196		
	134	631396		
	2	8232		
	1360	**63962800**		
	6	55136		
	1366	64017936		
	6	55152		
	1372	**64073088**		
	6			
	13780			
	4			
	13784			
	4			
	13788			
	4			
	13792			

First diminish the roots by 4. As the decimal part is now about to appear, attach ciphers to the coefficients of the transformed equation as explained in Ex. 2, Art. 108. Since the coefficient 130 is small in proportion to 5700, we may expect that the trial-divisor will give a good indication of the next figure. The figure to be adopted in every case as part of the root is ***that highest number which in the process of transformation will not change the sign of the absolute term.*** Here 2 is the proper figure In diminishing by 2 the roots of the transformed equation $x^3+130x^2+5700x-16000=0$,

the absolute term retains its sign (−4072). If we had adopted the figure 3, the absolute term would have become positive, the change of sign showing that we had gone beyond the root. We must take care that, after the first transformation (the reason of this restriction will appear in the next example), the absolute term preserves its sign throughout the operation. If we were to take by mistake, a number too small, the error would show itself, just as in ordinary division or evolution, by the next suggested number being greater than 9. Such a mistake however, will rarely be made. The error which is most common is to take the number too large, and this will show itself in the work by the change of sign in the absolute term. In the above work it is evident, without performing the fifth transformation, that the corresponding figure of the root is 4, so that the correct root to four decimal places is 4·2044.

2. The equation $x^4+4x^3-4x^2-11x+4=0$ has one root between 1 and 2; find its value correct to four decimal places.

1	4	−4	−11	4	(1·6369
	1	5	1	−10	
	5	1	−10	**−60000**	
	1	6	7	56976	
	6	7	**−3000**	**−90240000**	
	1	7	11496	72690561	
	7	**1400**	8496	**−175494390000**	
	1	516	14808	152131052016	
	80	1916	**23304000**	**−23363337984**	
	6	552	926187		
	86	2468	24230187		
	6	588	935601		
	92	**305600**	**25165788000**		
	6	3129	189387336		
	98	308729	25355175336		
	6	3138	189766488		
	1040	311867	**25544941824**		
	3	3147			
	1043	**31501400**			
	3	63156			
	1046	31564556			
	3	63192			
	1049	31627748			
	3	63228			
	10520	**31690976**			
	6				
	10526				
	6				
	10532				
	6				
	10538				
	6				

We see without completing the fifth transformation that 9 is the next figure of the root. The root is, therefore, 1·6369 correct to four decimal places.

The trial-divisor becomes effective after the second transformation, suggesting correctly the number 3, and all subsequent numbers. The first transformed equation has its last two terms negative. We may expect, therefore, that the influence of the preceding coefficients is greater than that of the trial-divisor, as in fact is here the case. The number 6, the second figure of the root, must be found by substitution. We have to determine what is the situation between 0 and 10 of the root of the equation

$$x^4+80x^3+1400x^2-3000x-60000=0.$$

A few trials show that 6 gives a negative, and 7 a positive result. Hence the root lies between 6 and 7; and 6 is the number of which we are in search. In the subsequent trials we take those greatest numbers 3, 6, 9, in succession, which allow the absolute term to retain its negative sign. In the first transformation, diminishing the roots by 1, there is a change of sign in the absolute term. The meaning of this is, that we have passed over a root lying between 0 and 1, for 0 gives a positive result, 4; and 1 gives a negative result, -6. In all subsequent transformations, so long as we keep below the root, the sign of the absolute term must be the same as the sign resulting from the substitution of 1. This supposes of course that no root lies between 1 and that of which we are in search. This supposition we have already made in the statement of the question In fact the proposed equation can have only two positive roots; one of them lies between 0 and 1 and, therefore, only one between 1 and 2.

When two roots exist between the limits employed in Horner's method, *i.e.*, when the equation has a pair of roots nearly equal, certain precautions must be observed which will form the subject of a subsequent Article.

3. Find the root of the preceding equation between 0 and 1 to four decimal places. Commence by multiplying by 10. The coefficients are then

$$1,\ 40,\ -400,\ -11000,\ 40000;$$

the trial-divisor becomes effective at once in consequence of the comparative smallness of the leading coefficients. The positive sign of the absolute term must be preserved throughout. [*Ans.* ·3373.

4. Find to three places of decimals the root situated between 9 and 10 of the equation

$$x^4-3x^2+75x-10000=0.$$ [*Ans.* 9·886.

[Supply the zero coefficient of x^3.]

In the examples hitherto considered the root has been found to a few decimal places only. We proceed now to explain a method by which, after three or four places of decimals have been evolved as above, several more may be correctly obtained with great facility by a contracted process.

110. Contraction of Horner's Process. In the ordinary process of Contracted Division, when the given figures are exhausted, in place of appending ciphers to the successive dividends, we cut off figures successively from the right of the divisor, so that the divisor itself becomes exhausted after a number of steps depending on the

number of figures it contains. The resulting quotient will differ from the true quotient in the last figure only, or at most in the last two figures. In Horner's contracted method the principle is the same. We retain those figures only which are effective in contributing to the result to the degree of approximation desired. When the contracted process commences, in place of appending ciphers to the successive co-efficients of the transformed equation in the way before explained, we cut off one figure from the right of the last co-efficient but one, two from the right of the last co-efficient but two, three from the right of the last co-efficient but three; and so on. The effect of this is to retain in their proper places the important figures in the work, and to banish altogether those which are of little importance.

The student will do well to compare the first transformation by the contracted process in the first of the following examples with the corresponding step in the second example of the last Article, where the transformation is exhibited in full. He will then observe how the leading figures (those which are most important in contributing to the result) coincide in both cases, and retain their relative places; while the figures of little importance are entirely dispensed with.

In addition to the contraction now explained, other abbreviations of Horner's process are sometimes recommended; but as the advantage to be derived from them is small, and as they increase the chances of error, we do not think it necessary to give any account of them. The contraction here explained is of so much importance in the practical application of Horner's method of approximation that no account of this method is complete without it.

Examples

1. Find the root between 1 and 2 of the equation in Ex. 2 of the last Article correct to seven or eight decimal places.

Assuming the result of the Example referred to, we shall commence the contracted process after the third transformation has been completed. The subsequent work stands as follows:—

1032	315014	25165788	−17549439	(1·636913575
	6	18936	15213090	
	3156	2535515	−2336349	
	6	18972	2301597	
	3162	2554487	−34752	
	6	285	25601	
	3168	255733	−9151	
		285	7680	
	31	256018	−1471	
			1280	
			−191	
			179	
			12	

Here the effect of the first cutting off of figures, namely, 8 from the second last co-efficient, 14 from the third last, and 052 from the fourth last, is to banish altogether the first co-efficient of the biquadratic. We proceed to diminish the roots by 6 as if the co-efficients 1, 3150, 2516578, −17549439 which are left were those of a cubic equation. In multiplying by the corresponding figure of the root the figures cut off should be multiplied mentally, and account taken of the number to be carried just as in contracted division.

After the diminution by 6 has been completed, we cut off again in the transformed cubic 7 from the last co-efficient but one, 68 from the last but two, and the first co-efficient disappears altogether. The work then proceeds as if we were dealing with the co-efficients 31, 255448, −2336349 of a quadratic. The effect of the next process of cutting off is to banish altogether the leading co-efficient 31. The subsequent work coincides with that of contracted division. When the operation terminates, the number of decimals in the quotient may be depended on up to the last two of three figures. The extent to which the evolution of the root must be carried before the contracted process is commenced depends on the number of decimal places required; for after the contraction commences we shall be furnished in addition to the figures already evolved, with as many more as there are figures, in the trial-divisor, less one.

2. Find to seven or eight decimal places the root of the equation

$$x^4-12x+7=0$$

which lies between 2 and 3.

This equation can have only two positive roots, one lies between 0 and 1, and the other between 2 and 3. For the evolution of the latter we have the following :—

0	0	−12	7	(2·0472755671
2	4	8	−8	
2	4	−4	−100000000	
2	8	24	83891456	
4	12	20000000	−16108544	
2	12	972864	15493401	
6	240000	20972864	−615143	
2	3216	985792	446262	
800	243216	21958656	−168881	
4	3232	17478	156226	
804	246448	2213343	−12655	
4	3248	17478	11159	
808	249696	2230821	−1496	
4	2496	49	1338	
812		223131	−158	
4		49	156	
816	24	223180	2	

On this we remark, that after diminishing the roots by 2, and multiplying the roots of the transformed equation by 10, we find that the trial-divisor 20000 will not "go into" the absolute term 10000; we put, therefore, zero in the quotient, and multiply again by 10, and then proceed as before.

3. Find the root of the same equation which lies between 0 and 1.
[*Ans.* ·593685829.

4. Find the positive root of the equation

$$x^3+24{\cdot}84x^2-67{\cdot}613x-3761{\cdot}2758=0.$$

When the co efficients of the proposed equation contain decimal points, it will be found that they soon disappear in the work in consequence of the successive multiplications by 10 after the decimal part of the root begins to appear. [*Ans.* 11·1973222.

5. Find the negative root of the equation

$$x^4-12x^2+12x-3=0$$

to seven places of decimals.

When a negative root has to be found, it is convenient to change the sign of x and find the corresponding positive root of the transformed equation.
[*Ans.* $-3{\cdot}9073785$.

111. Application of Horner's Method to Cases where Roots are nearly Equal. We have seen in Art. 107 that the method of approximation there explained fails when the proposed equation has two roots nearly equal. Examples of this nature are those which present most difficulties, both in their analysis (*see* Ex. 7, Art. 98) and in their solution. By Horner's method it is possible, with very little more labour than is necessary in other cases, to effect the solution of such equations. So long as the leading figures of the two roots are the same, certain precautions must be observed, which will be illustrated by the following examples. After the two roots have been separated, the subsequent calculation proceeds for each root separately, just as in the examples of the previous Articles. It is evident, from the explanation of the trial-divisor given in Art. 109, that for the same reason as that which explains the failure of Newton's method in the case under consideration (*see* Art. 107), it will not become effective till the first or second stage after the roots have been separated.

Examples

1. The equation

$$x^3-7x+7=0$$

has two roots between 1 and 2 (*see* Ex. 2, Art. 96); find each of them to eight decimal places.

Diminishing the roots by 1, we find that the transformed equation (after its roots are multiplied by 10), *viz.*,

$$x^3+30x^2-400x+1000=0,$$

must have two roots between 0 and 10. We find that these roots lie, one between 3 and 4, and the other between 6 and 7. The roots are now separated, and we proceed with each separately in the manner already explained. If the roots were not separated at this stage, we should find the leading figure common to the

two, and, having diminished the roots by it, find in what intervals the roots of the resulting equation were situated ; and so on. [*Ans.* 1·35689584 ; 1·69202147.

2. Find the two roots of the equation

$$x^3-49x^2+658x-1379=0$$

which lie between 20 and 30.

We shall exhibit the complete work of approximation to the smaller of the two roots to seven places ; and then make some observations which will be a guide to the student in all cases of the kind

1	−49	658	−1379	(23·2131277
	20	−580	1560	
	−29	78	**181**	
	20	−180	−180	
	−9	−**102**	**1000**	
	20	42	−992	
	11	−60	**8000**	
	3	51	−6739	
	14	−**900**	**1261000**	
	3	404	−1217403	
	17	−496	43597	
	3	408	−34183	
	200	−**8800**	9414	
	2	2061	−6786	
	202	−6739	2628	
	2	2062	−2372	
	204	−**467700**	256	
	2	61899	−236	
	2060	−405801	20	
	1	61908		
	2061	34389~~3~~		
	1	206		
	2062	−34183		
	1	206		
	20630	20~~6~~ − 3397~~7~~		
	3	4		
	20633	3393		
	3	4		
	20636	~~2~~ − 338~~9~~		
	3			
	206~~3~~~~9~~			

The diminution of the roots by 20 changes the sign of the absolute term. This is an indication that a root exists between 0 and 20, with which we are not at present concerned. The roots of the first transformed equation

$$x^3+11x^2-102x+181=0$$

are not yet separated, lying both between 3 and 4. The substitution of each of these numbers gives a positive result, so that we have not here the same criterion to guide us in our search for the proper figure as in former cases, *viz.*, a change of sign in the absolute term. We have, however, a different criterion which enables us to find by mere substitution the interval within which the roots lie If we diminish the roots of $x^3+11x^2-102x+181=0$ by 4, the resulting equation is $x^3+23x^2+34x+13=0$, which has no change of sign. Hence the two roots must lie between 0 and 4. If we diminish its roots by 3, the resulting equation (as in the above work) has the same number of changes of sign as the equation itself. Hence the two roots lie between 3 and 4. They are, therefore, not yet separated; and we proceed to diminish by 3. The next transformed equation

$$x^3+200x^2-900x+1000=0$$

is found in the same way to have both its roots between 2 and 3; the diminution by 2 leaving two changes of sign in the coefficients of the transformed equation (as in the above work), and the diminution by 3 giving all positive signs. So far, then, the two roots agree in their first three figures, *viz.*, 23·2. We diminish again by 2. The resulting equation $x^3+2060x^2-8800x+1261000=0$ has one root only between 1 and 2; 1 giving a positive, and 2 a negative result: its other root lies between 2 and 3; 3 giving a positive result. The roots are now separated. We proceed, as in the above work, to approximate to the lesser root, by diminishing the roots of this equation by 1; the trial divisor becoming effective at the next step. To approximate to the greater root, we must diminish by 2 the roots of the same equation, taking care that in the subsequent operations the negative sign, to which the previously positive sign of the absolute term now changes, is preserved. The second root will be found to be 23·2295212.

So long as the two roots remain together, a guide to the proper figure of the root may be obtained by dividing twice the last coefficient by the second last, or the second last by twice the third last. The reason of this is, that the proposed equation approximates now to the quadratic formed by the last three terms in each transformed equation; just as in previous cases, and in Newton's method, it approximated to the simple equation formed by the last two terms, this quadratic having the two nearly equal roots for its roots; and when the two roots of the equation $ax^2+bx+c=0$ are nearly equal, either of them is given approximately by $\dfrac{-2c}{b}$ or $\dfrac{-b}{2a}$. Thus, in the above example, the number 3 is suggested by $\dfrac{2\times 181}{102}$, and the number 2 by $\dfrac{2\times 1000}{900}$. In this way we can generally, at the first attempt, find the two integers between which the pair of roots lies. We shall have also an indication of the separation of the roots by observing when the numbers suggested in this way by the last three coefficients become different, *i.e.*, when $\dfrac{2c}{b}$ suggests a different number from $\dfrac{b}{2a}$.

3. Calculate to three decimal places each of the roots lying between 4 and 5 of the equation

$$x^4+8x^3-70x^2-144x+936=0.$$

[*Ans.* 4·242; 4·246.

4. Find the two roots between 2 and 3 of the equation

$$64x^3-592x^2+1649x-1445=0.$$

[*Ans.* The roots are both $=2·125$.

Here we find that the two roots are not separated at the third decimal place. When we diminish by 5, the absolute term vanishes, showing that 2·125 is a root; and proceeding with this diminution the second last coefficient also vanishes. Hence 2·125 is a double root.

When an equation contains more than two nearly equal roots, they can all be found by Horner's process in a manner similar to that now explained. Such cases are, however, of rare occurrence in practice. The principles already laid down will be a sufficient guide to the student in all cases of the kind.

112. Lagrange's Method of Approximation. Lagrange has given a method of expressing the root of a numerical equation in the form of a continued fraction. As this method is, for practical purposes, much inferior to that of Horner, we shall content ourselves with a brief account of it.

Let the equation $f(x)=0$ have one root, and only one root, between the two consecutive integers a and $a+1$. Substitute $a+\frac{1}{y}$ for x in the proposed equation. The transformed equation in y has one positive root. Let this be determined by trial to lie between the integers b and $b+1$. Transform the equation in y by the substitution $y=b+\frac{1}{z}$. The positive root of the equation in z is found by trial to lie between c and $c+1$. Continuing this process, an approximation to the root is obtained in the form of a continued fraction, as follows:—

$$a+\cfrac{1}{b+\cfrac{1}{c+1\ \ldots\ldots}}$$

Examples

1. Find in the form of a continued fraction the positive root of the equation

$$x^3-2x-5=0.$$

The root lies between 2 and 3.

To make the transformation $x=2+\frac{1}{y}$, we first employ the process of Art. 33 diminishing the roots by 2. We then find the equation whose roots are the reciprocals of the roots of the transformed.

The equation in y is in this way found to be

$$y^3-10y^2-6y-1=0.$$

This has a root between 10 and 11.

Make now the substitution $y=10+\frac{1}{z}$.

The equation in z is

$$61z^3-94z^2-20z-1=0.$$

This has a root between 1 and 2. Take $z=1+\frac{1}{u}$.

The equation in u is

$$54u^3+25u^2-89u-61=0,$$

which has a root between 1 and 2; and so on.

We have, therefore, the following expression for the root :-

$$x=2+\cfrac{1}{10+\cfrac{1}{1+\cfrac{1}{1+\dots}}}$$

2. Find in the form of a continued fraction the positive root of

$$x^3-6x-13=0.$$

$$\left[\textit{Ans.}\quad 3+\cfrac{1}{5+\cfrac{1}{1+\cfrac{1}{1+\dots}}}\right.$$

113. Numerical Solution of the Biquadratic. It is proper, before closing the subject of the solution of numerical equations, to illustrate the practical uses which may be made of the methods of solution of Chap. VI. Although, as before observed, the numerical solution of equations is in general best effected by the methods of the present chapter, there are certain cases in which it is convenient to employ the methods of Chap. VI for the resolution of the biquadratic. When a biquadratic equation leads to a reducing cubic which has a commensurable root, this root can be readily found, and the solution of the biquadratic completed. We proceed to solve a few examples of this kind, using Descartes' method (Art. 64), which will usually be found the most convenient in practice.

Examples

1. Resolve the quartic

$$x^4-6x^3+3x^2+22x-6$$

into quadratic factors.

Making the assumption of Art. 64, we easily obtain

$$p+p'=-3,\ q+q'+4pp'=3,\ pq'+p'q=11,\ qq'=-6,$$

also

$$\varphi=\frac{1}{2}-pp'=\frac{1}{4}(q+q'-1);$$

and, calculating and the equation for φ is

$$4\varphi^2-\frac{111}{4}\varphi-\frac{225}{8}=0.$$

Multiplying the roots by 4, we have, if $4\varphi=t$,

$$t^3-111t-450=0.$$

By the Method of Divisor this is easily found to have a root -6; hence $\varphi=-\frac{3}{2}$, giving

$$pp'=2,\ q+q'=-5.$$

From these, combined with the preceding equations, we get

$$p=-2,\ p'=-1,\ q=1,\ q'=-6.$$

When the values of q and q' are found, the equation giving the value of $pq'+p'q$ determines which value of q goes with p, and which with p', in the quadratic factors. The quartic is resolved, therefore, into the factors

$$(x^2-4x+1)(x^2-2x-6).$$

By means of the other two values of ϕ we can resolve the quartic into quadratic factors in two other ways; or we can do the same thing by solving the two quadratics already obtained.

2. Resolve into factors the quartic

$$f(x)\equiv x^4-8x^3-12x^2+60x+63.$$

The equation for ϕ is

$$4\phi^3-195\phi-475=0,$$

which is found to have a root $=-5$.

[*Ans.* $f(x)\equiv(x^2-2x-3)(x^2-6x-21)$.

3. Resolve into factors

$$f(x)\equiv x^4-17x^2-20x-6.$$

The reducing cubic is found to be

$$4\phi^3-\frac{217}{12}\phi+\frac{3185}{216}=0;$$

or, multiplying the roots by 6,

$$4t^3-651t+3185=0.$$

This has a root $=7$; hence $\phi=\frac{7}{6}$. [*Ans.* $f(x)\equiv(x^2+4x+2)(x^2-4x-3)$.

4. Resolve into factors

$$f(x)\equiv x^4-6x^3-9x^2+66x-22.$$

The reducing cubic is

$$4\phi^3-\frac{335}{4}\phi-\frac{897}{8}=0;$$

hence $\phi=-\frac{3}{2}$. [*Ans.* $f(x)\equiv(x^2-11)(x^2-6x+2)$.

5. Resolve into factors

$$f(x)\equiv x^4-8x^3+21x^2-26x+14.$$

[*Ans.* $f(x)\equiv(x^2-2x+2)(x^2-6x+7)$.

6. Resolve into factors

$$x^4+12x+3.$$

[*Ans.* $(x^2-x\sqrt{6}+3+\sqrt{6})(x^2+x\sqrt{6}+3-\sqrt{6})$.

7. Find the quadratic factors of

$$4x^4-8x^3-12x^2+84x-63=0,$$

and solve the equation completely (*see* Ex. 18, p. 26).

Ans. $[x^2-2x(2+\sqrt{7})+3\sqrt{7}][x^2-2x(2-\sqrt{7})-3\sqrt{7}]$.

Miscellaneous Examples

1. Find the positive root of

$$x^3-6x-13=0.$$

[*Ans.* 3·176814393.

2. Find the positive root of

$$x^3-2x-5=0$$

correct to eight or nine places. [*Ans.* 2·094551483.

3. The equation

$$2x^3-650\cdot 8x^2+5x-1627=0$$

has a root between 300 and 400 ; find it. [*Ans.* Commensurable root, 325·4.

4. Find the root between 20 and 30 of the equation

$$4x^3-180x^2+1896x-457=0.$$

[*Ans* 28·52127738.

5. Find to six places the root between 2 and 3 of the equation

$$x^3-49x^2+658x-1379=0.$$

[*Ans.* 2·557351.

6. Find to six places the root between 2 and 3 of the equation

$$x^4-12x^2+12x-3=0.$$

[*Ans.* 2·858083.

7. Find the positive root of the equation

$$x^3+2x^2-23x-70=0$$

correct to about ten decimal places. [*Ans.* 5·13457872528.

8. Find the cube root of 673373097125. [*Ans.* 8765.

9. Find the fifth root of 537824. [*Ans.* 14.

10. Find all the roots of the cubic equation

$$x^3-3x+1=0.$$

The equation $x^6+x^3+1=0$, of Ex. 7, p. 81, reduces to this.

[*Ans.* $-1\cdot 87938$, $0\cdot 34729$, $1\cdot 53209$.

The smaller positive root gives the solution of the problem. To divide a hemisphere whose radius is unity into two equal parts by a plane parallel to the base.

11. Find all the roots of the cubic

$$x^3+x^2-2x-1=0.$$

(*See* Ex. 1, p. 81.) [*Ans.* $-1\cdot 80194$, $-0\cdot 44504$, $1\cdot 24698$.

12. Find to five decimal places the negative root between -1 and 0 (*see* Ex. 3, p. 81) of the equation

$$x^5+x^4-4x^3-3x^2+3x+1=0.$$

[*Ans.* $-0\cdot 28463$.

13. Solve the equation

$$x^3-315x^2-19684x+2977260=0.$$

We find that there is a root here between 70 and 80. By Horner's process it is found to be 78. The depressed equation furnishes two roots, which, increased by 78, are the remaining roots of the cubic. [*Ans.* 78, 347, -110.

14. Find the two real roots of the equation

$$x^4+11727x+40385=0.$$

[*Ans.* 3·45592, 21·43067.

This equation is given by Mr. G. H. Darwin in a Paper *On the Precession of a Viscous Spheroid, and on the Remote History of the Earth. Phil. Trans.*, Part ii, 1879, p. 508. The roots are "the two values of the cube root of the earth's rotation for which the earth and moon move round as a rigid body."

15. Find all the roots of the cubic equation

$$20x^3-24x^2+3=0.$$

[*Ans.* $-0{\cdot}31469$, $0{\cdot}44603$, $1{\cdot}06865$.

This equation occurs in the solution by Professor Ball of a problem of Professor Townsend's in the *Educational Times* of Dec., 1878, to determine the deflection of a beam uniformly loaded and supported at its two ends and points of trisection.

16. Find the positive root of equation

$$14x^3+12x^2-9x-10=0.$$ [*Ans.* $0{\cdot}85906$.

The equations of this and the following example occur in the investigation of questions relative to beams supported by props.

17. Find the positive root of the equation

$$7x^4+20x^3+3x^2-16x-8=0.$$ [*Ans.* $0{\cdot}91336$.

18. Find to ten decimal places the positive root of the equation

$$x^5+12x^4+59x^3+150x^2+201x-207=0.$$ [*Ans.* $0{\cdot}6386058033$.

19. Find all the commensurable roots of

$$f(x)\equiv x^5+2x^4-36x^3-149x^2-232x-336=0,$$

and solve the equation completely. [*Ans.* $f(x)\equiv(x^2+x+3)(x+4)^2(x-7)$.

20. Solve similarly the equation

$$f(x)\equiv x^5-32x^4+116x^3-116x^2+115x-84=0.$$

[*Ans.* $f(x)\equiv(x^2+1)(x-1)(x-3)(x-28)$.

21. Find the condition that the quadratic Sturmian remainder of Ex. 3, Art. 99, should have its roots imaginary. [*Ans.* $HI+3aJ$ positive.

This condition is fulfilled when H and J are both positive (since then I must be positive, by the identity of Art. 37). It is, therefore, easily inferred that the biquadratic has no real roots when H and J are both positive (*cf.* Ex. 15, p. 173).

22. When the biquadratic has two roots equal to α, prove

$$a\alpha+b=\frac{-GI}{2HI-3aJ}.$$

23. If the equation $f(x)=0$ has all its roots real, prove that the equation $f(x)\,f''(x)-[f'(x)]^2=0$ has all its roots imaginary.

24. If an equation of any degree, arranged according to powers of x, have three consecutive terms in geometric progression, prove that its roots cannot be all real.

These three terms must be of the form $kx^r+k\alpha x^{r-1}+k\alpha^2x^{r-2}$. Let the equation be multiplied by $x-\alpha$. The resulting equation will have two consecutive terms absent, and must, therefore, have at least two imaginary roots; but all the roots of this equation except α are roots of the given equation.

25. If an equation has four consecutive coefficients in arithmetic progression, prove that its roots cannot be all real.

This can be reduced to the preceding example. Writing down four terms of the proper form, and multiplying by $x-1$, it readily appears that the resulting equation has three consecutive terms in geometric progression.

26. Calculate the first two of Sturm's remainders for a quintic wanting the second terms, *viz.*,

$$f(x)\equiv x^5+ax^3+bx^2+cx+d=0.$$

[*Ans.* $R_1\equiv-2ax^3-3bx^2-4cx-5d$,

$R_2\equiv Ax^2+Bx+C$,

where

$$A = 40ac-12a^3-45b^2,\ B = 50ad-8a^2b-60bc,\ C = -4a^2c-75bd.$$

Retaining this notation it is easy to calculate the coefficients D, E of the third remainder $R_3 = Dx+E$ in terms of a, b, c, d, A, B, C; and, finally, R_4 in terms of A, B, C, D, E.

27. Remove the second term from the general quintic written with binomial coefficients, and prove that the leading coefficients of the first two of Sturm's remainders for the resulting equation are

$$-H,\ -5HI+9a_0J.$$

28. Calculate the leading coefficients of the first two Sturmian remainders for an equation of the n^{th} degree wanting the second term, *viz.*,

$$x^n+ax^{n-2}+bx^{n-3}+cx^{n-4}+\text{etc.}=0.$$

No coefficients beyond those here given will enter into the required values; we readily find

$$R_1 = -2ax^{n-2}-3bx^{n-3}+4cx^{n-4}-\text{etc.}$$
$$R_2 = -[4(n-2)a^3-8nac+9nb^2]x^{n-3}+\text{etc.}$$

29. Remove the second term from the general equation of the n^{th} degree written with binomial coefficients, and prove that the leading coefficients of the first two Sturmian remainders of the resulting equation are

$$-H,\ -nHI+3(n-2)a_0J.$$

These expressions are easily derived from the preceding example by aid of the transformation of Art. 35; the values A_2, A_3, A_4 being given by equations

$$a_0A_2=H,\ a_0^3A_3=G,\ a_0^2A_4=a_0^2I-3H^2,$$

G^2 being placed by its value from the identity of Art. 37, and positive multipliers omitted.

30. Calculate Sturm's functions for Euler's cubic (*see* Art. 61).

We find after some reductions, and omitting positive factors,

$$f(x) = x^3+3Hx^2+3(H^2-\tfrac{1}{12}a^2I)x-\tfrac{1}{4}G^2,$$
$$f'(x) = x^2+2Hx+H^2-\tfrac{1}{12}a^2I,$$
$$R_1 = 2Ix+2HI-3aJ,$$
$$R_2 = I^3-27J^2.$$

All the conditions of Art. 68, with respect to the nature of the roots of the biquadratic, may be derived from these results, by the aid of Ex. 4, p. 102. And it will be observed that the conditions for reality of all the roots as given in Art. 100, as well as in the Article already referred to, are both obtained here together; for, in order that Euler's cubic should have all its roots real and positive, the substitution of 0 for x must give three changes of sign, and this requires that a^2I-12H^2 and $2HI-3aJ$ should be both negative.

CHAPTER XII

COMPLEX NUMBERS AND THE COMPLEX VARIABLE

114. Complex Numbers—Graphic Representation. In the foregoing chapters many examples have been met with of the occurrence among the solutions of numerical equations of quantities of the form $a+b\sqrt{-1}$, involving the extraction of the square root of a negative number. Such an expression, consisting of a positive or negative real units, and b positive or negative imaginary units, is called a *complex number* (*see* Art. 15). The imaginary unit $\sqrt{-1}$ is denoted for brevity by i. Real and purely imaginary numbers are both included in the expression $a+ib$, the former being obtained when $b=0$, and the latter, when $a=0$. Complex numbers may be submitted to all the ordinary rules of arithmetical calculation; and in the result of any such calculation integral powers of i beyond the first can always be reduced by the relation $i^2=-1$.

We proceed to explain a mode of representing complex numbers geometrically, which will be found very convenient in the treatment of functions involving quantities of this kind.

The expression $a+ib$ may be written in the form

$$\mu(\cos \alpha + i \sin \alpha),$$

where

$$\mu=\sqrt{a^2+b^2},\ \cos \alpha=\frac{a}{\mu},\ \sin \alpha=\frac{b}{\mu}.$$

The quantity μ is called the *modulus*, and the angle α the *amplitude*, of the complex number $a+ib$. The modulus is always taken positively, the negative sign of the radical corresponding to an increase of the amplitude by π.

Let rectangular axes OX, OY (Fig. 7) be taken, and a point A such that $XOA=\alpha$, and $OA=\mu$. We have then $OM=\mu \cos \alpha=a$, and $AM=\mu \sin \alpha=b$. The expression $a+ib$ may, therefore, be represented graphically by the right line drawn from O to a point in a plane whose co-ordinates referred to the fixed axes are a, b; the distance OA of this point from the origin being equal to the

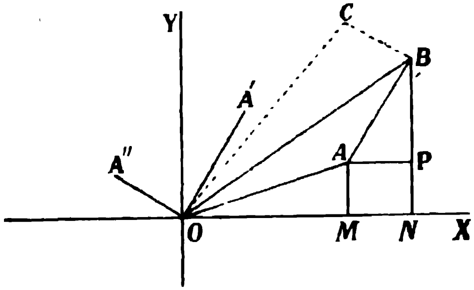

Fig. 7.

modulus, and the angle XOA equal to the amplitude of the complex number.

The magnitude of a complex quantity is estimated the magnitude of its modulus. When the complex quantity vanishes (that is, when a and b separately vanish), its modulus vanishes; and, conversely, when the modulus vanishes, since then $a^2+b^2=0$, a and b must separately vanish, and, therefore, the complex quantity itself. Two such quantities, $a+ib$ and $a'+ib'$, are equal when $a=a'$ and $b=b'$, *i.e.*, when the moduli are equal and when the amplitudes either are equal or differ by a multiple of 2π.

In what follows we shall, for brevity, represent the modulus and amplitude of $a+ib$ by the notation

$$mod. \, (a+ib), \, amp. \, (a+ib).$$

115. Complex Numbers.—Addition and Subtraction. Let a second complex number $a'+ib'$ be represented by the right line OA', so that

$$OA'=mod. \, (a'+ib'), \, XOA'=amp.(a'+ib').$$

We proceed to determine the mode of representing the sum

$$a+ib+a'+ib'.$$

Writing this sum in the form $a+a'+i(b+b')$, we observe, in accordance with the convention of Art. 114, that it will be represented by the line drawn from the origin to the point whose co-ordinates are $a+a'$, $b+b'$. To find this point, draw AB parallel and equal to OA'; since AP, BP are equal to a', b', B is the required point, and we have

$$OB=mod. \, \{a+a'+i(b+b')\}, \, XOB=amp. \, \{a+a'+i(b+b')\}$$

To add two complex numbers, therefore, we draw OA to represent one of them; and, at its extremity, AB to represent the second (that is, so that its length is equal to the modulus, and the angle it makes with OX equal to the amplitude, of the second); then OB represents the sum of the two complex numbers.

Since OB is not greater than $OA+AB$, it follows that *the modulus of the sum of two complex numbers is less than* (*or at most equal to*) *the sum of their moduli.*

This mode of representation may be extended to the addition of any number of such quantities. Thus, to add a third $a''+ib''$ represented by OA'', we draw BC parallel and equal to OA'', and join OC. Then OC represents the sum of the three, OA, OA', OA''. It is evident also that we may conclude in general that the *modulus of the*

sum of any number of complex quantities is less than (or at most equal to) the sum of their moduli.

Subtraction can be represented in a similar way. Since OB represents the sum of OA and OA', OA will represent the difference of OB and OA'. To subtract two complex numbers, therefore, we draw at the extremity of the line representing the first a line parallel and equal to the second, but in an opposite direction (*i.e.*, a direction which makes with OX an angle greater by π than the amplitude of the second). We join O to the extremity of this line to find the right line which represents the difference of the two given complex numbers.

116. Multiplication and Division. To multiply the two complex numbers $a+ib$, $a'+ib'$, we write them in the form

$$a+ib \equiv \mu\,(\cos\alpha + i\sin\alpha),\ a'+ib' \equiv \mu'\,(\cos\alpha' + i\sin\alpha').$$

We have then, by De Moivre's theorem,

$$(a+ib)\,(a'+ib') \equiv \mu\mu'\,\{\cos(\alpha+\alpha') + i\sin(\alpha+\alpha')\},$$

which proves that *the product of two complex numbers is a complex number, whose modulus is the product of the two moduli, and whose amplitude is the sum of the two amplitudes.*

In the same way it appears that the product of any number of such factors is a complex quantity, whose modulus is the product of all the moduli, and whose amplitude is the sum of all the amplitudes.

To divide $a+ib$ by $a'+ib'$, we have similarly

$$\frac{a+ib}{a'+ib'} = \frac{\mu}{\mu'}\,\{\cos(\alpha-\alpha') + i\sin(\alpha-\alpha')\}$$

which proves that *the quotient of two complex numbers is a complex number, whose modulus is the quotient of the two moduli, and whose amplitude is the difference of two amplitudes.*

It was assumed in the proof of the theorem of Art. 16 that when a product of any number of factors (real or imaginary) vanishes, one of the factors must vanish. This is evident when the factors are all real. From what is above proved the same conclusion holds when the factors are complex; for, in order that the modulus of the product may vanish, one of its factors must vanish, and, therefore, the complex quantity of which that factor is the modulus.

117. Other Operations on Complex Numbers. From the foregoing propositions it follows that any integral power of a complex number, *e.g.*, $(a+ib)^m$, can be expressed in the form $A+iB$, where A and B are real. And, more generally, if in any rational integral function

$$a_0z^n + a_1z^{n-1} + a_2z^{n-2} + \ldots + a_{n-1}z + a_n,$$

whose coefficients are complex (including real) numbers, a complex quantity $a+ib$ be substituted for z, the result can be expressed in the standard form $A+iB$.

It is not proposed in the present chapter to discuss any functions of complex numbers beyond the rational integral function of the kind hitherto treated in this work. It is easy, however, to show, by the aid of De Moivre's theorem, that the remaining processes of numerical calculation—powers with fractional or complex exponents, logarithms, and powers whose base and exponent are both complex—reproduce in every case a complex number as result. This is expressed by saying that complex numbers form a system or group complete in themselves.

118. The Complex Variable. In the earlier chapters of the present work the variation of a polynomial was studied corresponding to the passage of the variable through real values from $-\infty$ to $+\infty$; and the mode of representing by a figure the form of the polynomial was explained. Such a mode of treatment is only a particular case of a more general inquiry. Given a polynomial, rational and integral in z, whose coefficients are numbers real or complex, viz.,

$$f(z) \equiv a_0 z^n + a_1 z^{n-1} + a_2 z^{n-2} + \ldots + a_{n-1} z + a_n,$$

we may study its variations corresponding to the different values of z, where z has the complex form $x+iy$, and where x and y both take all possible real values. This form $x+iy$ is called the *complex variable*. All possible *real* values of the variable are of course included in the values of $x+iy$, being those values which arise by varying x and putting $y=0$. In accordance with the principles of Art. 114 we may represent the complex variable $x+iy$ by the line OP (Fig. 8) drawn from a fixed origin O to the point whose co-ordinates are x, y. Or we may say, $x+iy$ is represented by the point P. Thus all possible values of $x+iy$ will be represented by all the points in a plane. Since, for any particular values of z, $f(z)$ takes the form $A+iB$ (Art. 117), the values of $f(z)$ may be represented in a similar manner by points in a plane. We confine ourselves in the present Article to the representation of the variable $x+iy$ itself. We conceive the variation of $x+iy$ to take place in continuous manner; for example, by the motion of the point x, y, along a curve. If OP and OP' repre-

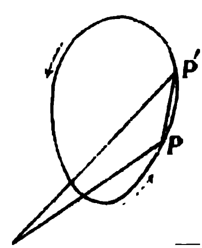

Fig. 8.

sent two consecutive values of the variable, we write the corresponding values $x+iy$, $x'+iy'$ as follows :

$$z\equiv x+iy\equiv r(\cos\theta+i\sin\theta),\ z'\equiv x'+iy'\equiv r'(\cos\theta'+i\sin\theta').$$

Since OP' represents the sum of OP and PP' (Art. 115), it follows that PP' represents the increment of z ; and if $z'=z+h$, h may be written in the form

$$h\equiv\rho(\cos\phi+i\sin\phi),$$

where $\rho=PP'$, and ϕ is the angle PP' makes with OX.

The variation of the modulus of z is $OP'-OP$ or $r'-r$; the variation of the amplitude of z is $P'OP'$ or $\theta'-\theta$; the variation of z itself is h or ρ $(\cos\phi+i\sin\phi)$, as just explained.

Let the point be supposed to describe a closed curve. When it returns to its original position P, the modulus takes again its original value ; and the amplitude takes its original value if the point O is exterior to the curve, or is increased by 2π if O is interior to the curve.

If the complex variable describes the same line in two opposite directions, the variations of its amplitude are equal and of opposite signs, *i.e.*, the total variation is nothing. From this we can derive a property of the variation of the amplitude of the complex variable, which will be found of importance in our succeeding investigations.

Let a plane area be divided into any number of parts by lines BD, AF, EC, etc. (Fig. 9); then *the variation of the amplitude relatively to the perimeter of the whole area is equal to the sum of its variations relatively to the perimeters of the partial areas* : all the areas being supposed to be described by the variable moving in the same sense. This is evident ; for when the point is made to describe all the partial areas in the same sense, each of the internal dividing lines will be described twice, the two descriptions being in opposite directions ; and the external perimeter will be described once ; hence the total variation of the amplitude relatively to the dividing lines vanishes ; and the variation relatively to the external perimeter alone remains. Take, for example, the areas ABF, AFD in the figure. When the point describes these areas in the sense indicated by the arrows, the total variation relatively to the line AF vanishes.

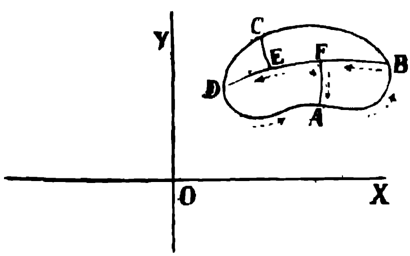

Fig. 9.

119. Continuity of a Function of the Complex Variable. Suppose the complex variable z, starting from a fixed value z_0, to receive a small increment $h \equiv \rho(\cos\phi + i \sin\phi)$; we have then, if $f(z)$ be the given function, replacing x by z in the expansion of Art. 6,

$$f(z) = f(z_0 + h) = f(z_0) + f'(z_0)h + \frac{f''(z_0)}{1.2}h^2 + \text{etc.},$$

and the increment of $f(z)$, being equal to $f(z_0+h) - f(z_0)$, is

$$f'(z_0)h + \frac{f''(z_0)}{1.2}h^2 + \frac{f'''(z_0)}{1.2.3}h^3 + \text{etc...}$$

In this expression the co-efficients of the powers of h are all complex expressions of the usual form ; and if their moduli be a, b, c, etc., the moduli of the successive terms are $a\rho$, $b\rho^2$, $c\rho^3$, etc.; and since by Art. 115, the modulus of a sum is less than the sum of the moduli, if follows that the modulus of the increment of $f(z)$ is less than

$$a\rho + b\rho^2 + c\rho^3 + \text{ etc.}$$

Now a value may be assigned to ρ (Art. 5) such that for it or any smaller value the value of this expression will be less than any assigned quantity. It follows that to an infinitely small variation of the complex variable (*viz.*, one whose modulus is infinitely small) corresponds an infinitely small variation of the function ; in other words, *the function varies continuously at the same time as the complex variable itself.*

120. Variation of the Amplitude of $f(z)$ corresponding to the description of a small Closed Curve by the Complex Variable. Corresponding to a continuous series of values of z we have a continuous series of values of $f(z)$, which can be represented, like the

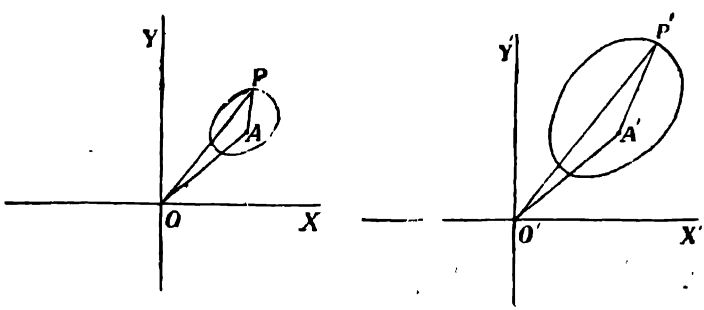

Fig. 10.

values of z itself, by points in a plane. We represent these series of points by two figures (Fig. 10) side by side, which, to avoid confusion,

may be supposed to be drawn on different planes. To each point P, representing $x+iy$, corresponds one determinate point P' representing $f(z)$. When P describes a continuous curve, P' describes also a continuous curve ; and when P returns to its original position after describing a closed curve, P' returns also to its original position.

Our present object is to discuss the variation of the amplitude of $f(z)$ corresponding to the description of a small closed curve by P. Let A be any determinate point whose co-ordinates are x_0, y_0, *i.e.*, $z_0=x_0+iy_0$. We divide the discussion into two cases :—

(1) When x_0+iy_0 is not a root of $f(z)=0$, *i.e.*, when $f(z_0)$ is different from zero.

(2) When x_0+iy_0 is a root of $f(z)=0$, or $f(z_0)=0$.

(1) In the first case, to the point A corresponds a point A' representing the value of $f(z_0)$, and $O'A'$ is different from zero. Let $z=z_0+h$, where $h\equiv \rho\ (\cos\phi+i\sin\phi)$; and suppose P, which represents z, to describe a small closed curve round A. Let P' represent $f(z)$; then $A'P'$ represents the increment of $f(z)$ corresponding to the increment AP of z. By the previous Article it appears that values so small may be assigned to ρ, that the modulus of the increment of $f(z)$, namely $A'P'$, may be always less than the assigned qnantity $O'A'$; hence P may be supposed to describe round A a closed curve so small that the corresponding closed curve described by P' will be exterior to O'. It follows, by Art. 118, that *corresponding to the description by P of a small closed curve, which does not contain a point satisfying the equation $f(z)=0$, the total variation of the amplitude of $f(z)$ is nothing.*

(2) In the second case, suppose x_0+iy_0 is a root of the equation $f(z)=0$ repeated m times, and let

$$f(z)\equiv(z-z_0)^m\psi(z)\ ;$$

then

$$f(z)=h^m\psi(z)=\rho^m\ (\cos m\phi+i\sin m\phi)\ \psi\ (z).$$

In this case $O'A'=0$; and when P describes a closed curve round A, P' returns to its original position, and the amplitude of $f(z)$ will be increased by a multiple of 2π, which may be determined as follows :—From the above equation we have

$$amp.\ f(z)=m\varphi+amp.\ \psi\ (z)\ ;$$

and the increment of $amp.\ f(z)$ will be obtained by adding the increment of $m\phi$ to the increment of $amp.\ \psi(z)$. Now the latter increment is nothing by (1), since the curve described by P may be supposed

to contain no root of $\psi(z)=0$; and since the increment of ϕ is 2π in one revolution of P, the increment of $m\phi$ is $2m\pi$. It follows that *when P describes a small closed curve containing a root of the equation $f(z)=0$, repeated m times, the amplitude of $f(z)$ is increased by $2m\pi$.*

121. Cauchy's Theorem. When z describes the same line in a plane in two opposite directions, $f(z)$ describes the corresponding line in its plane in two opposite directions, and the *amp.* $f(z)$ undergoes equal and opposite variations. It follows that, if any plane area be divided into its parts, as in Art. 118, the variation of the *amp.* $f(z)$ corresponding to the description in the same sense by z of all the partial areas, is equal to the variation of *amp.* $f(z)$ corresponding to the description by z of the external perimeter only. Now let any closed perimeter in the plane XY be described; and suppose, in the first place, that it contains no point which satisfies the equation $f(z)=0$. It can be broken up into a number of small areas, with respect to each of which the conclusions of (1), Art. 120, hold; and by what has been just proved, it follows that the variation of *amp.* $f(z)$ corresponding to the description by z of the closed perimeter is nothing. Suppose, in the second place, that the closed perimeter contains a point which is a root of the equation $f(z)=0$ repeated m times. Let a small closed curve $PQRS$ be described round this point. The variation of *amp.* $f(z)$ corresponding to the description by z of the whole perimeter, is equal to the sum of its variations corresponding to the description of the areas $ABCPSR$, $CDARQP$, $PQRS$. The two former variations vanish by what is above proved; and the latter is, by (2), Art. 120, equal to $2m\pi$. The total variation, therefore, of $f(z)$ is $2m\pi$. Similarly, if the area includes additional points which correspond to roots repeated m', m'', etc., times, the total variation $=2(m+m'+m''+\text{etc.})\pi$. Hence we derive the following theorem due to Cauchy :—

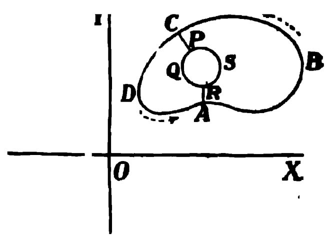

Fig 11.

The number of roots of any polynomial, comprised within a given plane area, is obtained by dividing by 2π the total variation of the amplitude of this polynomial corresponding to the complete description by the complex variable of the perimeter of the area.

122. Number of Roots of the General Equation. We are enabled by means of the principles established in the preceding

Articles to prove the theorem contained in Arts. 15 and 16; namely, *Every rational and integral equation of the n^{th} degree has n roots real or imaginary.*

Let

$$f(z)\equiv a_0z^n+a_1z^{n-1}+a_2z^{n-2}+\ldots+a_{n-1}z+a_n$$

be a rational and integral function of z. Without making any supposition as to the existence of roots of $f(z)=0$ further than that $f(z)$ cannot vanish for any infinite values of the variable, we can suppose z to describe in its plane a circle so large that no root exists outside of it. If, then,

$$f(z)=z^n(a_0+a_1z'+a_2z'^2+\ldots.+a_nz'^n)$$
$$=z^n\phi\ (z'),\ \text{where}\ z'=\frac{1}{z},$$

z', whose modulus is the reciprocal of the modulus of z, will describe a small circle containing a portion of the plane corresponding to the part outside of the circle described by z; and no root of $\phi(z')=0$ will be included within this small circle. Hence, corresponding to the description of the whole circle by z, the variation of *amp.* $f(z')=0$, and, therefore,

$$\text{variation of } amp.\ f(z)=\text{variation of } amp.\ z^n;$$

and if $\quad z=r\ (\cos\theta+i\sin\theta)$ or $z^n=r^n\ (\cos n\theta+i\sin n\theta)$,

θ is increased by 2π, and, thereforel, *amp.* z^n is increased by $2n\pi$.

It follows from Cauchy's theorem, Art. 121, that the number of roots comprised within the circle described by z, *i.e.*, the total number of roots of the equation $f(z)=0$, is n; and the theorem is proved.

The proposition whose proof was deferred in Art. 15 is thus shown to be an immediate consequence of Cauchy's theorem, which may, therefore, be regarded as the fundamental proposition of the theory of equations. It is proper to observe, however, that the theorem of Art. 15, *viz.*, that every numerical equation has a numerical root, can be proved directly, and independently of Cauchy's theorem, by aid of the principles contained in Art. 119 and the preceding Articles, as we proceed now to show.

123. Second Proof of Fundamental Theorem. If possible let there be no value of z which makes $f(z)$ vanish; and let the value z_0, represented by A, Fig. 10, correspond to the nearest possible position, A', of P' to the origin O'. It is proposed to show that such a direction may be given to the increment h as to bring P' into a posi-

tion nearer to the origin than A' We have the following expansion (Art. 119) :—

$$f(z_0+h)=f(z_0)+f'(z_0)h+\frac{f''(z_0)}{1\,.\,2}h^2+\ldots+a_0h^n.$$

By hypothesis $f(z_0)$ does not vanish ; but one or more of the derived functions, $f'(z_0)$, etc., may do so. Let the first of these which does not vanish be $f_m(z_0)$, and let us suppose

$$\frac{f_m(z_0)}{1\,.\,2\,.\,3\ldots m}=\mu_m(\cos\alpha_m+i\sin\alpha_m),$$

with corresponding expressions for the coefficients which follow. Collecting all the terms which contain powers of h beyond h^m into one complex expression, we may write

$$f(z_0+h)=f(z_0)+\mu_m\rho^m[\cos(m\phi+\alpha_m)+i\sin(m\phi+\alpha_m)]$$
$$+\mu(\cos\xi+i\sin\xi),$$

where, by the proposition of Art. 115,

$$\mu<\mu_{m+1}\rho^{m+1}+\mu_{m+2}\rho^{m+2}+\ldots+\mu_n\rho^n.$$

It is easily inferred from the theorem of the Art. 5 that such a value may be given to ρ as to make $\mu<\mu_m\rho^m$. Now the direction of the increment h can be so selected, *viz.*, from the equation $m\varphi+\alpha_m=X'O'A'+\pi$ (fig. 10), as to bring P', in virtue of the second expression in the value of $f(z_0+h)$, through a distance $\mu_m\rho^m$ nearer to the origin in the direction $A'O'$. Let S be the point on the line $O'A'$ to which P' is brought in this way. The effect of the last expression in the value of $f(z_0+h)$ is to move P' from S to a point T at a distance $ST=\mu$; and whatever the direction of this movement, *i.e.*, whatever the amplitude ξ, $O'T$ is $<O'A'$, since $ST<SA'$. We have proved, therefore, that A' is not the nearest possible position of P' with reference to the origin ; and in the same manner it may be shown that no other value different from zero can be the least possible value of the modulus of $f(z)$.

In the proof here given it is only shown that the equation must have a root, and the precise number of roots is not determined, as it is in the proof derived from Cauchy's theorem ; but it is proved that one root at least must exist, the proof can be easily completed by the method of Art. 16.

It is important to observe that when $f'(z_0)$ does not vanish, from any particular point z_0 the limiting value of the ratio of the increment of $f(z_0)$ to h is the constant $f'(z_0)\equiv\mu_1(\cos\alpha_1+i\sin\alpha_1)$. It is easily inferred that the two increments are inclined at a constant

angle, and their moduli are in a constant ratio. This is usually expressed by saying that the figures described by P and P' are similar in their infinitely small parts.

The student is referred to Note C at the end of the volume for some further observations on the subject of this Article.

124. Determination of Complex Numerical Roots.—Solution of the Cubic. Little attention has been given by writers on the Theory of Equations to the actual determination of the complex numerical roots of equations; nor is it easy to give any account suitable to an elementary text-book of general methods in existence for this purpose. Theoretically the problem presents no difficulty; for if the real and imaginary parts of $f(x+iy)$ be equated separately to zero, and from the two resulting equations one of the variables eliminated, an equation is obtained from which a real value of the remaining one can be calculated by Horner's process. It will be found, however, that this method is of little practical value.*

We confine ourselves in this and the following Articles to cubic and biquadratic equations with real numerical coefficients, and exhibit the calculation in these instances in what appears to be the simplest form for practical purposes. Let the equation

$$f(x) \equiv x^3 + px^2 + qx + r = 0$$

be proposed for solution. The roots may be assumed to be α, $h+k$, $h-k$, of which α is real. The character of the remaining roots will appear in the process of calculation; k being determined from its square, which may turn out to be either positive or negative. No preliminary analysis of the equation is necessary. If $h+k$ be substituted for x, and the sums of the even and odd powers of k equated separately to zero, as in Ex. 26, p. 124, we find immediately the equation

$$-k^2 = f'(h) = 3h^2 + 2ph + q.$$

We get also, by the elimination of k, a cubic equation for the determination of h; but there will be no occasion to form this equation, since h is best got from the relation $\alpha + 2h = -p$, α having

*The student desirous of information as to the attempts of mathematicians in the direction of the calculation of imaginary roots of numerical equations may refer to the following works:—Lagrange's *Traité de la Résolution des Equations numériques*; Murphy's *Theory of Algebraical Equations*; *Allgemeine Auflösung der Zahlen-Gleichungen*, by Simon Spitzer (Wien, 1851); *Die Auflösung der höheren numerischen Gleichungen*, by P.C. Jelinek (Leipzig, 1865); *A method for calculating simultaneously all the Roots of an Equation*, by Emory M'Clintock (*American Journal of Mathematics*, vol. xvii, Nos. 1 and 2); and *Méthode pratique pour la Résolution numérique complète des Equations algébriques ou transcendantes*, by M.E. Carvallo (Paris, 1896).

been calculated in the first instance in the usual way by Horner's method.

It will be necessary finally to calculate k, and with it the remaining two roots, whether real or imaginary. For this purpose the following mode of procedure will be found convenient :—The value of $\Sigma f'(\alpha)$ in terms of coefficients is p^2-3q, *viz* ,

$$f'(\alpha)+f'(h+k)+f'(h-k)=p^2-3q\;;$$

also

$$f'(h+k)+f'(h-k)=2f'(h)+6k^2\;;$$

whence immediately,

$$f'(\alpha)+4k^2=p^2-3q,$$

from which k^2 can be determined with very little labour, since *the numerical value of $f'(\alpha)$ can be written down from the second last coefficient in the final transformation in the work of Horner's process already completed.* The character of the remaining two roots will depend on the sign of the number so found ; and the roots themselves will be determined by taking the positive square roots of this number.

Examples

1. Solve the equation

$$x^3+2x^2-23x-70=0.$$

First calculate the real positive root, completing four transformations by Horner's method, and obtaining for the final transformed equation the following coefficients :—

$$1,\quad 17402,\quad 76609868,\quad -44341896.$$

Remembering that the roots have been three times multiplied by 10, we find the values of $f(\alpha)$ and $f'(\alpha)$ by cutting off nine figures from the right in the former case, and six in the latter, and supplying the decimal point. It is well to carry the approximation a couple of steps further by the contracted method, and thus get a more accurate value of $f'(\alpha)$. We find, in this way,

$$f'(\alpha)=76{\cdot}6286.$$

Subtracting this number from p^2-3q, which is equal to 73, we find

$$4k^2=-3{\cdot}6286.$$

Since this is negative, we have proved that the remaining roots are imaginary. From the ascertained value of α, *viz.*, $6{\cdot}13457$, the value of h is found immediately to be $-3{\cdot}5672$, and dividing $3{\cdot}6286$ by 4, and taking its square root, we have finally the two complex roots of the equation as follows :—

$$-3{\cdot}5672\pm0{\cdot}9524\sqrt{-1}.$$

2. Solve completely Newton's cubic (*see* Art. 107), *viz.*,

$$x^3-2x-5=0.$$

Completing four transformations by Horner, and proceeding as in the former example, we find $\alpha=2{\cdot}09455$, and

$$f'(\alpha)=11{\cdot}16078\;;$$
$$k^2=-1{\cdot}290195,$$

and the remaining two roots (thus proved imaginary) are found to be

$$-1\cdot04727 \pm 1\cdot13594\sqrt{-1}.$$

3. Find the remaining two roots of the example of Art. 109, p. 187, *viz.*

$$x^3+x^2+x-100=0.$$

We find $f'(\alpha)=64\cdot0841$, $k^2=-16\cdot52102$, and the required roots are

$$-2\cdot6322 \pm 4\cdot0646\sqrt{-1}.$$

4. Solve the equation

$$20x^3-24x^2+3=0.$$

Dividing by 20, and applying Horner's process to find the root of the equation $x^3+1\cdot2x^2+\cdot15=0$ lying between 0 and 1, we find $\alpha=0\cdot4460366$, and $f'(\alpha)=-0\cdot47364$. We have, therefore,

$$4k^2=p^2-3q-f'(\alpha)=1\cdot44+0\cdot47364;$$

hence $k^2=\cdot47841$, and the remaining two roots are real. We find $h=\cdot37698$; and adding and subtracting k, the other roots are found to be $1\cdot06865$ and $-0\cdot31469$ (*cf.* Ex. 15, p. 200).

5. Solve completely Lagrange's cubic

$$x^3-7x+7=0.$$

Change the signs of all the roots, and calculate the positive root α between 3 and 4 of the transformed equation $f(x)=0$, thus obtaining $\alpha=3\cdot0489173$, and $f'(\alpha)=20\cdot88737$; hence $k^2=\cdot0281575$, and $k=\cdot1678$. Also $h=-1\cdot524458$; whence the values of $h+k$ and $h-k$; and changing the signs of all the roots thus found, the roots of the given equation are

$$-3\cdot0489, \qquad 1\cdot3566, \qquad 1\cdot6922. \qquad (\textit{Cf.}\ \text{Ex. 1, Art. 111})$$

The examples given are sufficient to show in what way this process may be used to solve a given numerical cubic, without any previous examination of the character of its roots. The amount of work required decide in this way whether the two remaining roots are real or imaginary is usually very little greater than is required in the application of Sturm's theorem; and the additional labour necessary for the actual determination of the roots is extremely small

We proceed now to biquadratic equations.

125. Solution of the Biquadratic. When a biquadratic equation has real roots (two or four), it can be solved in a manner analogous to that employed in the preceding Article. In some examples the existence of a real root can be at once recognized; and when such is the case, the following process for the complete solution of the equation can be used with advantage. Let the proposed equation be

$$f(x) \equiv x^4+px^3+qx^2+rx+s=0,$$

and its real roots α, β; the remaining two roots may be represented by $h+k$ and $h-k$, no assumption being made as to the character of the latter pair. Let α and β be both calculated by Horner's process, and the numerical values of $f'(\alpha)$ and $f'(\beta)$ determined at the same

time, as in the preceding Article. Now, if $h+k$ be substituted for x in $f(x)$, and the method of solution of Ex. 26, p. 124, employed, we find, without difficulty,

$$-k^2 = \frac{6f'(h)}{f'''(h)} = \frac{4h^3+3ph^2+2qh+r}{4h+p}.$$

Again, we have, as is easily proved,

$$f'(\alpha)+f'(\beta)+f'(h+k)+f'(h-k) = -p^3+4pq-8r,$$

and

$$f'(h+k)+f'(h-k) = 2f'(h)+f'''(h)k^2,$$

whence immediately

$$-4k^2(4h+p) = f'(\alpha)+f'(\beta)+p^3-4pq+8r.$$

This formula can be used for the calculation of k^2, the value of h having been previously ascertained from the equation $\alpha+\beta+2h = -p$ by means of the calculated values of α and β. The second pair of roots will be real or imaginary according as the resulting value of k^2 is positive or negative.

Examples

1. Solve completely the equation

$$x^4-3x^3+7x^2-10x+1=0.$$

It is at once apparent that a real root exists between 0 and 1. There must, therefore, be a second, which is found to lie between 1 and 2. By Horner's process we find

$$\alpha = 0{\cdot}107767, \qquad \beta = 1{\cdot}923262,$$
$$f'(\alpha) = -8{\cdot}59078, \qquad f'(\beta) = 12{\cdot}09133$$

whence we have

$$f'(\alpha)+f'(\beta)+p^3-4pq+8r = -19{\cdot}49945.$$

Also, from the values of α, β and p, $h = 0{\cdot}484485$, and $4h+p = -1{\cdot}06206$; therefore,

$$-4k^2 = \frac{19{\cdot}49945}{1{\cdot}06206}.$$

It is now proved that the remaining two roots are imaginary and their values can be ascertained by calculating k from this formula. Logarithmic Tables will assist in the calculation. The roots are found to be

$$0{\cdot}4845 \pm 2{\cdot}1424\sqrt{-1}.$$

2. Solve completely the equation of Ex. 2, Art. 110, *viz.*,

$$x^4-12x+7=0.$$

We find

$$\alpha = 0{\cdot}59368, \qquad \beta = 2{\cdot}04727$$
$$f'(\alpha) = -11{\cdot}1635, \qquad f'(\beta) = 22{\cdot}3180;$$

whence the pair of imaginary roots

$$-1{\cdot}32048 \pm 2{\cdot}0039\sqrt{-1}.$$

3. Solve the equation

$$2x^4-13x^2+10x-19=0.$$

There must be two real roots : one (α) positive, and the other (β) negative. Divide by 2, and write the equation as follows :—

$$f(x) \equiv x^4 - 6\cdot 5x^2 + 5x - 9\ 5 = 0.$$

When β is calculated in the usual way by first changing the signs of the roots of $f(x)=0$, it is to be observed that, in order to get the value of $f'(\beta)$, we must change the signs of the second last coefficient supplied by the final transformation in Horner's process. We find

$$\alpha = 2\cdot 45733, \qquad \beta = -3\cdot 03055,$$
$$f'(\alpha) = 32\cdot 409, \qquad f'(\beta) = -66\cdot 936\ ;$$

whence

$$-4k^2 = \frac{5\cdot 473}{1\cdot 1464},$$

and the imaginary roots

$$0\ 2866 \pm 1\cdot 0924\sqrt{-1}.$$

4. Solve the equation

$$x^4 - 80x^3 + 1998x^2 - 14937x + 5000 = 0.$$

There is clearly a real root between 0 and 1, and a second is easily seen to lie between 12 and 13 (*see* Ex. 4, Art. 93). We find

$$\alpha = 0\cdot 35098, \qquad \beta = 12\cdot 75644,$$
$$f'(\alpha) = -13564, \qquad f'(\beta) = 5286\cdot 7\ ;$$

whence

$$4k^2 = \frac{413\cdot 3}{53\cdot 785}.$$

The remaining two roots, therefore, are real, and are easily found to be $32\cdot 0602$ and $34\cdot 8322$.

All the roots of this equation have been calculated by Horner's method by Young (*Analysis and Solution of Cubic and Biquadratic Equations*, pp. 216–221). Our last two roots agree, to the number of places here given, with the values arrived at by him.

126. Solution of Biquadratic continued. When the roots of a biquadratic equation are all imaginary, the mode of solution of the preceding Article of course fails. In this case, and in general, whatever be the nature of the roots, the following method may be used :—Let the equation, first deprived of its second term, be written in the form

$$f(x) \equiv x^4 + qx^2 + rx + s = 0.$$

The roots of this may be assumed to be $h \pm k$, $-h \pm k'$, no assumption being made as to their character, which will depend on the signs of k^2 and k'^2 when calculated. Substituting $h+k$ for x, and proceeding as before, we find

$$-k^2 = \frac{6f'(h)}{f'''(h)} = \frac{4h^3 + 2qh + r}{4h},$$

whence

$$-4k^2 = 4h^2 + 2q + \frac{r}{h},$$

from which k can be found when h is known. When k is eliminated between the two equations of Ex. 26, p. 124 ; the sextic in h reduces to the cubic

$$y^3+2qy^2+(q^2-4s)y-r^2=0,$$

of which $4h^2$ is a root. This cubic must have one positive root : the remaining two may be both positive, both negative, or both imaginary, according to the nature of the roots of the given biquadratic. The equation is, in fact, Euler's reducing cubic (with roots multiplied by 4) for the biquadratic under consideration (*see* Ex. 4, p. 102). Let the positive root of the cubic be calculated by Horner's process (if the three are positive, any one of them will do). Thus $4h^2$ is determined, and from it h ; and the full solution of the proposed biquadratic equation is given by the two formulae

$$h\pm\sqrt{-\frac{1}{4}\left(4h^2+2q+\frac{r}{h}\right)},\quad -h\pm\sqrt{-\frac{1}{4}\left(4h^2+2q-\frac{r}{h}\right)}.$$

Examples

1. Give the complete solution of the equation

$$x^4+x+10=0.$$

This equation is used by Murphy (*Theory of Equations*, p. 125) to illustrate his proposed method of determining the imaginary roots of equations by means of recurring series. We find readily the reducing cubic

$$y^3-40y-1=0,$$

and, by Horner's process, the positive root 6·3370184 ; hence the value of h^2, and from it $h=\pm 1{\cdot}2586$. We find then $\frac{r}{h}=\pm 0{\cdot}7945$, according as the positive or negative sign of h is used. In either case the quantity under the square root is negative, and the roots are, therefore, all imaginary. They are easily found to be

$$1{\cdot}2586\pm 1{\cdot}3352\sqrt{-1},\qquad -1{\cdot}2586\pm 1{\cdot}1771\sqrt{-1}.$$

2. Solve the equation

$$x^4+9x^2-6x+5=0.$$

This example is treated by Spitzer (*Allgemeine Auflösung der Zahlen-Gleichungen* p. 15). The reducing cubic is

$$y^3+18y^2+61y-36=0,$$

whose positive root is found to be 0·51094249 ; hence $h=\pm 0{\cdot}35740$. The numerical value of r divided by h is found to be 16·7878 ; and whether h be taken with positive or negative sign, the quantity under the square root is negative, and, therefore, all the roots imaginary. The four roots are—

$$0{\cdot}3574\pm 0{\cdot}6563\sqrt{-1},\qquad -0{\cdot}3574+2{\cdot}9706\sqrt{-1}.$$

3. Solve the equation

$$x^4-2x^3-7x^2+10x+10=0.$$

To remove the second term, multiply the roots by 2, and then diminish roots by 1. The reducing cubic of the transformed equation is easily found to be

$$y^3-68y^2+320y-256=0.$$

Divide the roots of this by 10 and find immediately that the transformed equation has a root between 6 and 7, which is found by Horner's process to be 6·29838. Hence $4h^2=62·9838$, and $h=\pm 3·968$. Whether h is taken positively or negatively, it is found that the quantity under the square root is a positive number, and, therefore, all the roots are real in this case. We find $4k^2=9·04840$, $4k'^2=0·98400$; hence $k=\pm 1·504$, $k'=\pm 0·496$; whence, taking account of the two transformations which were made in removing the second term, we have the four roots as follows :—

$$2·732, \qquad 2·236, \qquad -0·732, \qquad -2·236.$$

The results, in this instance, can be readily verified, for it is easily seen that the given function is the product of the factors x^2-5 and x^2-2x-2 (compare also Ex. 5, p. 169).

4. Solve the equation

$$x^4-7x^3+7x^2-7x+7=0.$$

This example is discussed by Jelinek (*Die Auflösung der höheren numerischen Gleichungen*, p. 29). To remove the second term, multiply the roots by 4, and then diminish by 7. We find in this way

$$x^4-182x^2-1624x-3059=0,$$

whose reducing cubic is

$$y^3-364y^2+45360y-2637376=0.$$

To find the situation of the positive roots, it is well to divide the roots by 100, when it readily appears that the transformed equation has a root between 2 and 3. By Horner's process it is found to be 2·0591; whence $4h^2=205·91$, and $h=\pm 7·17$. When h is taken positively, the quantity under the square root is found to be positive; hence two real roots; and when it is taken negatively, the quantity under the square root is negative and gives a pair of imaginary roots. Taking account of the two transformations employed to remove the second term, we find the four roots of the proposed equation as follows :—

$$5·993, \qquad 1·091, \qquad -0·042\pm 1·033\sqrt{-1}.$$

5. Solve the equation

$$x^4-80x^3+1998x^2-14937x+5000=0.$$

This is Young's equation, already solved in the preceding Article. We repeat its solution here by the method of the present Article, in order that the student may have an opportunity of comparing the amount of labour required in the two methods. When the second term is easily removed (as is the case in the present instance), or when the second term is already absent in an equation, it will usually be found that the method of the present Article is the more expeditious of the two. Diminishing the roots by twenty, we find

$$x^4-402x^2+983x+25460=0,$$

whose reducing cubic is

$$y^3-804y^2+59764y-966289=0.$$

We get, by Horner's process, $4h^2=723·21038$, and, therefore, $h=\pm 13·4462$. The quantity under the square root is found to be positive whichever sign of h is taken, and for the four roots we have the two formulae

$$-h\pm\sqrt{38·47390}, \qquad h\pm\sqrt{1·92090};$$

hence, adding 20 to each root, we have the four roots of the proposed equation as follows :—

$$12·7565, \qquad 0·3511, \qquad 34·8321, \qquad 32·0603.$$

6. Solve completely the equation of Ex. 4, p. 133, *viz.*,

$$x^4-3x^2+75x-10000=0.$$

The roots are

$$9{\cdot}8860, \qquad -10{\cdot}2609, \qquad 0{\cdot}18748 \pm 9{\cdot}027\sqrt{-1}.$$

7. Solve completely the equation Ex. 2, p. 168 *viz.*,

$$x^4-4x^3-3x+23=0.$$

The roots are

$$3{\cdot}7853, \qquad 2{\cdot}0526, \qquad -0{\cdot}9189 \pm 1{\cdot}4545\sqrt{-1}.$$

8. Solve the equation of Ex. 4, p. 173, *viz.*,

$$x^4+3x^3-x^2-3x+11=0.$$

Multiply the roots by 4, and remove the second term. When Horner's method is applied to the reducing cubic, it is found that the latter equation has a commensurable root $=180$; hence $h=3\sqrt{5}$. The solution is easily completed, and the four imaginary roots expressed as follows :—

$$-\frac{3}{4}+\frac{3}{4}\sqrt{5} \pm \frac{1}{4}\sqrt{-10-2\sqrt{5}}, \quad -\frac{3}{4}-\frac{3}{4}\sqrt{5} \pm \frac{1}{4}\sqrt{-10+2\sqrt{5}}.$$

9. Find the imaginary roots of equation of Ex. 14, p. 199 *viz.*,

$$x^4-11727x+40385=0.$$

[*Ans.* $-12{\cdot}4433 \pm 19{\cdot}7596\sqrt{-1}$.

NOTES

NOTE A

ALGEBRAIC SOLUTION OF EQUATIONS

The solution of the quadratic equation was known to the Arabians, and is found in the works of Mohammed Ben Musa and other writers published in the ninth century. In a treatise on Algebra by Omar Alkhayyami, which belongs probably to the middle of the eleventh century, is found a classification of cubic equation, with methods of geometrical construction, but no attempt at a general solution. The study of Algebra was introduced into Italy from the Arabian writers by Leonardo of Pisa early in the thirteenth century; and for a long period the Italians were the chief cultivators of the science. A work styled *L'Arte Maggiore*, by Lucas Paciolus (known as Lucas de Burgo), was published in 1494. This writer adopts the Arabic classification of cubic equations, and pronounces their solution to be as impossible in the existing state of the science as the quadrature of the circle. At the same time he signalizes this solution as the problem to which the attention of mathematicians should be next directed in the development of the science. The solution of the equation $x^3+mx=n$ was effected by Scipio Ferreo; but nothing more is known of his discovery than that he imparted it to his pupil Florido in the year 1505. The attention of Tartaglia was directed to the problem in the year 1530, in consequence of a question proposed to him by Còlla, whose solution depended on that of a cubic of the form $x^3+px^2=q$. Florido, learning that Tartaglia had obtained a solution of this equation, proclaimed his own knowledge of the solution of the form $x^3+mx=n$. Tartaglia, doubting the truth of his statement, challenged him to a disputation in the year 1535, and in the meantime himself discovered the solution of Ferreo's form $x^3+mx=n$. This solution depends on assuming for x an expression $\sqrt[3]{t}-\sqrt[3]{u}$, consisting of the difference of radicals; and, in fact, constitutes the solution usually known as Cardan's. Tartaglia continued his labours, and discovered rules for the solution of the various forms of cubics included under the classification of the Arabic writers.

Cardan, anxious to obtain a knowledge of these rules, applied to Tartaglia in the year 1539, but without success. After many solicitations Tartaglia imparted to him a knowledge of these rules, receiving from him, however, the most solemn and sacred promises of secrecy. Regardless of his promises, Cardan published in 1545 Tartaglia's rules in his great work styled *Ars Magna*. It had been the intention of Tartaglia to publish his rules in a work of his own. He commenced the publication of this work in 1556, but died in 1559, before he had reached the consideration of cubic equations. As his work, therefore, contained no mention of his own rules, these rules came in process of time to be regarded as the discovery of Cardan, and to be called by his name.

The solution of equations of the fourth degree was the next problem to engage the attention of Algebraists ; and here, as well as in the case of the cubic, the impulse was given by Colla, who proposed to the learned the solution of the equation $x^4+6x^2+36=60x$. Cardan appears to have made attempts to obtain a formula for equations of this kind ; but the discovery was reserved for his pupil Ferrari. The method employed by Ferrari was a transformation of such a nature as to make both sides of the equation perfect squares, a new unknown quantity being introduced which is itself determined by an equation of the third degree. It is, in fact, virtually the method of Art. 63. This solution is sometimes ascribed to Bombelli, who published it in his treatise on Algebra in 1579. The solution known as Simpson's, which was published much later (about 1740), is in no respect essentially different from that of Ferrari In the year 1637 appeared Descartes' treatise, containing many improvements in algebraical science, the chief of which are his recognition of the negative and imaginary roots of equations, and his "Rule of Signs." His expression of the biquadratic as the product of two quadratic factors, although deducible immediately from Ferrari's form, was an important contribution to the study of this quantic. Euler's Algebra was published in 1770. His solution of the biquadratic (*see* Art. 61) is important, inasmuch as it brings the treatment of this form into harmony with that of the cubic by means of the assumed irrational form of the root. The methods of Descartes and Euler were the result of attempts made to obtain a general algebraic solution of equations. Throughout the eighteenth century many mathematicians occupied themselves with this problem ; but their labours were unsuccessful in the case of equations of a degree higher than the fourth.

In the solutions of the cubic and biquadratic obtained by the older analysts we observe two distinct methods in operation; the first, illustrated by the assumptions of Tartaglia and Euler, proceeding from an assumed explicit irrational form of the root; the other, seeking by the aid of a transformation of the given function to change its factorial character, so as to reduce it to a form readily resolvable. In Art. 55 these two methods are illustrated; together with a third, the conception of which is to be traced to Vandermonde and Lagrange, who published their researches about the same time, in the years 1770 and 1771. The former of these writers was the first to indicate clearly the necessary character of an algebraical solution of any equation, *viz.*, that it must, by the combination of radical signs involved in it, represent any root indifferently when the symmetric functions of the roots are substituted for the functions of the coefficients involved in the formula (*see* Art. 101). His attempts to construct formulae of this character were successful in the case of the cubic and biquadratic, but failed in the case of the quintic. Lagrange undertook a review of the labours of his predecessors in the direction of the general solution of equations, and traced all their results to one uniform principle. This principle consists in reducing the solution of the given equation to that of an equation of lower degree, whose roots are linear functions of the roots of the given equation and of the roots of unity. He shows also that the reduction of a quintic cannot be effected in this way, the equation on which its solution depends being of the sixth degree.

All attempts at the solution of equations of the fifth degree having failed, it was natural that mathematicians should inquire whether any such solution was possible at all. Demonstrations have been given by Abel and Wantzel (*see* Serret's *Cours d' Algebre Supérieure*, Art. 516) of the impossibility of resolving algebraically equations unrestricted in form, of a degree higher than the fourth. A transcendental solution, however, of the quintic has been given by M. Hermite, in a form involving elliptic integrals. Among other contributions to the discussion of the quintic since the researches of Lagrange, one of leading importance is its expression in a trinomial form by means of the Tschirnhausen transformation. Tschirnhausen himself had succeeded in the year 1683, by means of the assumption $y=P+Qx+x^2$, in the reduction of the cubic and quartic, and had imagined that a similar process might be applied to the general equation. The reduction of the quintic to the trinomial form was published by Mr. Jerrard in his *Mathematical Researches*, 1832-1835, and has

been pronounced by M. Hermite to be the most important advance in the discussion of this quantic since Abel's demonstration of the impossibility of its solution by radicals. In a Paper published by the Rev. Robert Harley in the *Quarterly Journal of Mathematics*, vol. vi., p. 38, it is shown that this reduction had been previously effected, in 1786, by a Swedish mathematician named Bring. Of equal importance to Bring's reduction is Dr. Sylvester's transformation, by means of which the quintic is expressed as the sum of three fifth powers—a form which gives great facility to the treatment of this quantic. Other contributions which have been made in recent years towards the discussion of quantics of the fifth and higher degrees have reference chiefly to the invariants and covariants of these forms. For an account of these researches, additional to what will be found in the second volume of this work, the student is referred to Clebsch's *Theorie der binären algebraischen Formen*, and to Salmon's *Lessons Introductory to the Modern Higher Algebra*.

NOTE B

SOLUTION OF NUMERICAL EQUATIONS

The first attempt at a general solution by approximation of numerical equations was published in the year 1600, by Vieta. Cardan had previously applied the rule of "false position" (called by him "regula aurea") to the cubic; but the results obtained by this method were of little value. It occurred to Vieta that a particular numerical root of a given equation might be obtained by a process analogous to the ordinary processes of extraction of square and cube roots; and he inquired in what way these known processes should be modified in order to afford a root of an equation whose co-efficients are given numbers. Taking the equation $f(x)=Q$, where Q is a given number, and $f(x)$ a polynomial containing different powers of x, with numerical co-efficients, Vieta showed that, by substituting in $f(x)$ a known approximate value of the root, another figure of the root (expressed as a decimal) might be obtained by division. When this value was obtained, a repetition of the process furnished the next figure of the root; and so on. It will be observed that the principle of this method is identical with the main principle involved in the methods of approximation of Newton and Horner (Arts. 107, 108). All that has been added since Vieta's time to this mode of solution of numerical equations is the arrangement of the calculation so as to afford facility and security in the process of evolution of the root. How great has been the improvement in this respect may be judged of by an observation in Montucla's *Histoire des Mathématiques*, vol. i, p. 603, where, speaking of Vieta's mode of approximation, the author regards the calculation (performed by Wallis) of the root of a biquadratic to eleven decimal places as a work of the most extravagant labour. The same calculation can now be conducted with great ease by anyone who has mastered Horner's process explained in the text.

Newton's method of approximation was published in 1669; but before this period the method of Vieta had been employed and simplified by Harriot, Oughtred, Pell, and others. After the period of Newton, Simpson and the Bernoullis occupied themselves with the same problem. Daniel Bernoulli expressed a root of an equation in the form of a recurring series, and a similar expression was given by

Euler; but both these methods of solution have been shown by Lagrange to be in no respect essentially different from Newton's solution (*Traité de la Résolution des Equations numériques*). Up to the period of Lagrange, therefore, there was in existence only one distinct method of approximation to the root of a numerical equation; and this method, as finally perfected by Horner in 1819, remains at the present time the best practical method yet discovered for this purpose.

Lagrange, in the work above referred to, pointed out the defects in the methods of Vieta and Newton. With reference to the former he observed that it required too many trials; and that it could not be depended on, except when all the terms on the left-hand side of the equation $f(x)=Q$ were positive. As defects in Newton's method he signalized—first, its failure to give a commensurable root in finite terms; secondly, the insecurity of the process which leaves doubtful the exactness of each fresh correction; and lastly, the failure of the method in the case of an equation with roots nearly equal. The problem Lagrange proposed to himself was the following:—"Etant donnée une équation numérique sans aucune notion préalable de la grandeur ni de l'espece de ses racines, trouver la valeur numérique exacte, s'il est possible, ou aussi approchée qu'on vondra de chacune de ses racines."

Before giving an account of his attempted solution of this problem, it is necessary to review what had been already done in this direction, in addition to the methods of approximation above described. Harriot discovered in 1631 the composition of an equation as a product of factors, and the relations between the roots and coefficients. Vieta had already observed this relation in the case of a cubic; but he failed to draw the conclusion in its generality, as Harriot did. This discovery was important, for it led to the observation that any integer root must be a factor of the absolute term of an equation; and Newton's Method of Divisors for the determination of such roots was a natural result. Attention was next directed towards finding limits of the roots, in order to diminish the labour necessary in applying the method of divisors as well as the methods of approximation previously in existence. Descartes, as already remarked, was the first to recognise the negative and imaginary roots of equations; and the inquiry commenced by him as to the determination of the number of real and of imaginary roots of any given equation was continued by Newton, Stirling, De Gua, and others.

Lagrange observed that, in order to arrive at a solution of the problem above stated, it was first necessary to determine the number of the real roots of the given equation, and to separate them one from another. For this purpose he proposed to employ the equation whose roots are the squares of the differences of the roots of the given equation. Waring had previously, in 1762, indicated this method of separating the roots ; but Lagrange observes (*Equations numériques*, Note iii) that he was not aware of Waring's researches when he composed of his own memoir on this subject. It is evident that when the equation of differences is formed, it is possible, by finding an inferior limit to its positive roots, to obtain a number less than the least difference of the real roots of the given equation. By substituting in succession numbers differing by this quantity, the real roots of the given equation will be separated. When the roots are separated in this way, Lagrange proposed to determine each of them by the method of continued fractions, explained in the text (Art. 112). This mode of obtaining the roots escapes the objections above stated to Newton's method, inasmuch as the amount of error in each successive approximation is known ; and when the root is commensurable, the process ceases of itself, and the root is given in a finite form. Lagrange gave methods also of obtaining the imaginary roots of equations, and observed that if the equation had equal roots, they could be obtained in the first instance by methods already in existence.

Theoretically, therefore, Lagrange's solution of the problem which he proposed to himself is perfect. As a practical method, however, it is almost useless. The formation of the equation of differences for equations of even the fourth degree is very laborious, and for equations of higher degrees becomes well-nigh impracticable. Even if the more convenient modes of separating the roots discovered since Lagrange's time be taken in conjunction with the rest of his process, still this process is open to the objection that it gives the root in the form of a continued fraction, and that the labour of obtaining it in this form is greater than the corresponding labour of obtaining it by Horner's process in the form of a decimal. It will be observed also that the latter process, in the perfected form to which Horner has brought it, is free from all the objections to Newton's method above stated.

Since the period of Lagrange, the most important contributions to the analysis of numerical equations, in addition to Horner's improvement of the methods of approximation of Vieta and Newton,

are those of Fourier, Budan, and Sturm. The researches of Budan were published in 1807 ; and those of Fourier in 1831, after his death. There is no doubt, however, that Fourier had discovered before the publication of Budan's work the theorem which is ascribed to them conjointly in the text. The researches of Sturm were published in 1835. The methods of separation of the roots proposed by these writers are fully explained in Chapter X. By a combination of these methods with that of Horner, we have now a solution of Lagrange's problem far simpler than that proposed by Lagrange himself. And it appears impossible to reach much greater simplicity in this direction. In extracting a root of an equation, just as in extracting an ordinary square or cube root, labour cannot be avoided ; and Horner's process appears to reduce this labour to a minimum. The separation of the roots also, especially when two or more are nearly equal, must remain a work of more or less labour. This labour may admit of some reduction by the consideration of the functions of the coefficients which play so important a part in the theory of the different quantics. If, for example, the functions H, I and J, are calculated for a given quartic, it will be possible at once to tell the character of the roots (*see* Art. 68). Mathematicians may also invent in process of time some mode of calculation applicable to numerical equations analogous to the logarithmic calculation of simple roots. But at the present time the most perfect solution of Lagrange's problem is to be sought in a combination of the methods of Sturm and Horner.

All that has been said applies only to the real roots of numerical equations. We have referred, in a foot-note on p. 212, to the chief works in which attempts have been made to give general methods of calculation of the imaginary or complex roots ; and in Arts. 124, 125, we have shown how these roots may be calculated most expeditiously in the case of equations of the third and fourth degrees with real numerical coefficients.

NOTE C

THE PROPOSITION THAT EVERY EQUATION HAS A ROOT

It is important to have a clear conception of what is proved, and what it is possible to prove, in connexion with the proposition discussed in Arts. 122, 123. If in the equation

$$a_0x^n+a_1x^{n-1}+\ldots+a_n=0$$

the coefficients a_0, a_1, ..., a_n are used as mere algebraical symbols without any restriction—that is to say, if they are not restricted to denote either real numbers or complex *numbers* of the form treated in Chapter XII—then, with reference to such an equation it is not proved, and there exists no proof, that every equation has a root. The proposition which is capable of proof is that, in the case of any rational integral equation of the n^{th} degree, whose coefficients are all complex (including real) numbers, there exist n complex numbers which satisfy this equation ; so that, using the terms *number* and *numerical* in the wide sense of Chapter XII, the proposition under consideration might be more accurately stated in the form —*Every numerical equation of the n^{th} degree has n numerical roots.*

As regards this proposition, there appears little doubt that the most direct and scientific proof is one founded on the treatment of imaginary expressions or complex numbers of the kind considered in Chapter XII. The first idea of the representation of complex numbers by points in a plane is due to Argand, who in 1806 published anonymously in Paris a work entitled *Essai sur une manière de representer les quantités imaginaires dans les constructions géométriques.* This writer some years later gave an account of his researches in Gergonne's *Annales.* Notwithstanding the publicity thus given by Argand to his new methods, they attracted but little notice, and appear to have been discovered independently several years later by Warren in England and Mourey in France. These ideas were developed by Gauss in his works published in 1831 ; and by Cauchy, who applied them to the proof of the important theorem of Art. 121. With reference to the proposition now under discussion, the proof which we have given in Art. 123 is a modification of a proof found in Argand's original memoir, and reproduced by Cauchy in his *Exercices d' Analyse.* A proof in many respects similar was given by Mourey.

Before the discovery of the geometrical treatment of complex numbers, several mathematicians occupied themselves with the problem of the nature of the roots of equations. An account of their researches is given by Lagrange in Note IX of his *Equations numériques*. The inquiries of these investigators, among whom we may mention D'Alembert, Descartes, Euler, Foncenex, and Laplace, referred only to equations with rational coefficients; and the object in view was, assuming the existence of factors of the form $x-\alpha$, $x-\beta$, etc., to show that the roots α, β, etc., were all either real or imaginary quantities of the type $a+b\sqrt{-1}$; in other words, that the solution of an equation with real numerical coefficients cannot give rise to an imaginary root of any form except the known form $a+b\sqrt{-1}$, in which a and b are real quantities. For the proof of this proposition, the method employed in general was to show that, in case of an equation whose degree contained 2 in any power k, the possibility of its having a real quadratic factor might be made to depend on the solution of an equation whose degree contained 2 in the power $k-1$ only, and by this process to reduce the problem finally to depend on the known principle that every equation of odd degree with real coefficients has a real root. Lagrange's own investigations on this subject, given in Note X of the work above referred to, related, like those of his predecessors, to equations with rational coefficients, and are founded ultimately on the same principle of the existence of a real root in an equation of odd degree with real coefficients.

As resting on the same basis, *viz.*, the existence of a real root in an equation of odd degree, may be noticed two recently published methods of considering this problem—one by the late Professor Clifford (*see* his *Mathematical Papers* p. 20, and *Cambridge Philosophical Society's Proceedings*, II, 1876), and the other by Mr. Malet (*Transactions of the Royal Irish Academy*, vol. xxvi, p. 453, 1878). Starting with an equation of the $2m^{th}$ degree, both writers employ Sylvester's dialytic method of elimination to obtain an equation of the degree $m(2m-1)$ on whose solution the existence of a root of the proposed equation is shown to depend; and since the number $m(2m-1)$ contains the factor 2 once less often than the number $2m$, the problem is reduced ultimately to depend, as in the methods above mentioned, on the existence of a root in an equation of odd degree. The two equations between which the elimination is supposed to be effected are of the degrees m and $m-1$; and the only difference between the two modes of proof consists in the manner of arriving at these equations. In Mr. Malet's methods they are found by means of

a simple transformation of the proposed equation ; while Professor Clifford obtains them by equating to zero the coefficients of the remainder when the given polynomial is divided by a real quadratic factor. The general forms of these coefficients will be found among the Miscellaneous Examples appended to the Chapter on Determinants in the second volume of this work ; and it will be readily observed that the elimination of β from the equations so obtained will furnish an equation in α of the degree $m(2m-1)$. (*See* Ex. 38, p. 34. vol. II.)

INDEX TO VOLUME I

[The figures appended refer to the pages.]